Design of Robust Fractional Order $PI^{\alpha}D$ Speed Control for Two-inertia System

Chengbin Ma* Student Member
Yoichi Hori** Member

This article deals with the speed control of two-inertia system by fractional order $PI^{\alpha}D$ controller which means the order of I controller can not only be integer but also be any real number. The significance of fractional order control is that it is a generalization and "interpolation" of the classical integer order control theory, which can achieve more adequate modeling and more direct design of robust control system. However, most of fractional order control researches were originated and concentrated on chemical process control, while in motion control the research is still in a primitive stage. In this article, a frequency-band fractional order $PI^{\alpha}D$ controller is proposed to the two-inertia speed control, which is a basic control problem in motion control. A frequency-band broken-line approximation method is also introduced to realize the $PI^{\alpha}D$controller that has a satisfactory accuracy in frequency domain. The superior robustness performances of $PI^{\alpha}D$ control system against input torque saturation and load inertia variation are shown by comparison of its experimental time responses with integer order PID control's. This robustness highlights the promising aspects of applying fractional order control to motion control.

Keywords: Fractional Order Control, Robust, Speed Control, Two-inertia System

1. Introduction

The concept of expanding derivatives and integrals to fractional (non-integer) orders is by no means new. In fact, this concept had a firm and long standing theoretical foundation. *Leibniz* mentioned it in a letter to *Hospital* over three hundred years ago (1695) and the earliest more or less systematic studies seem to have been made in the beginning and middle of the 19th century by *Liouville*(1832), *Holmgren*(1864) and *Riemann*(1953) (1). At the same time, Fractional Order Control (FOC) was introduced by *Tustin* for the position control of massive objects half century ago, where the actuator saturation requires the sufficient phase margin around and below the critical point (2).

However, the concept of FOC, in which the controlled systems and/or controllers are described by fractional order differential equations, was not widely incorporated into control engineering mainly due to the conceptually difficult idea of taking fractional order, the existence of so few physical applications and the limited available computational power at that time (3).

In last few decades, researchers pointed out that fractional order differential equations could model various real materials more adequately than integer order ones and provide an excellent tool for the description of dynamical processes (4) (5). Those fractional order models need the corresponding fractional order controllers to be proposed and evoked resurging interest to various applications of FOC (6) (7) (8). With the remarkable progress of computation power, modeling and realization of the FOC systems also became possible and much easier than before.

The significance of FOC is that it is a generalization and "interpolation" of classical integral order control theory, which could lead to more adequate modeling of dynamic processes and more direct design of robust control systems. However, most of these works were originated and concentrated on chemical process control while in motion control, the research is still in a primitive stage (4) (8).

This article is organized as follows: in section II, Mathematical aspects of FOC are mentioned; in section III, an integer order PID controller is designed for the two-inertia speed control; in section IV, a frequency-band $PI^{\alpha}D$ controller is proposed and its broken-line realization method is also introduced; in Section V, Experimental results are presented to show the robustness of proposed fractional order $PI^{\alpha}D$ controller. Finally, in section VI, conclusions are drawn.

2. Mathematical Aspects

2.1 Mathematical Definitions

The mathematical definition of fractional derivatives and integrals has been the subject of several different approaches (1) (4). The most frequently encountered definition is called Riemann-Liouville definition, in which the fractional order integrals are defined as

$$_{t_0}D_t^{-\alpha} = \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t-\xi)^{\alpha-1} f(\xi) d(\xi) \quad \cdots\cdots\cdots (1)$$

* Department of Electrical Engineering, School of Engineering, University of Tokyo
** Information & System Division, Electrical Control System Engineering, Institute of Industrial Science, University of Tokyo

while the definition of fractional order derivatives is

$$ {}_{t_0}D_t^{\alpha} = \frac{d^{\gamma}}{dt^{\gamma}} \left[{}_{t_0}D_t^{-(\gamma-\alpha)} \right] \cdots\cdots\cdots\cdots\cdots\cdots\cdots (2) $$

where

$$ \Gamma(x) \equiv \int_0^{\infty} y^{x-1} e^{-y} dy \cdots\cdots\cdots\cdots\cdots\cdots\cdots (3) $$

is the Gamma function, a and t are limits and α ($\alpha > 0$ and $\alpha \in R$) is the order of the operation. γ is an integer that satisfies $\gamma - 1 < \alpha < \gamma$.

The other approach is Grünwald-Letnikov definition:

$$ {}_{t_0}D_t^{\alpha} = \lim_{\substack{h \to 0 \\ nh = t - t_0}} h^{-\alpha} \sum_{r=0}^{n} (-1)^{\alpha} f(t - rh) \cdots\cdots (4) $$

Where the binomial coefficients ($r > 0$)

$$ \left(\begin{array}{c} \alpha \\ 0 \end{array} \right) = 1, \left(\begin{array}{c} \alpha \\ r \end{array} \right) = \frac{\alpha(\alpha-1)\ldots(\alpha-r+1)}{r!} (5) $$

2.2 Laplace and Fourier Transforms The Laplace transform of the Riemann-Liouville fractional order derivative with order $\alpha > 0$ is [1] [4]

$$ L\{{}_0D_t^{\alpha}\} = s^{\alpha}F(s) - \sum_{j=0}^{n-1} s^j \left[{}_0D_t^{\alpha-j-1} f(0) \right] \cdots (6) $$

where $(n-1) \leq \alpha < n$. If

$$ {}_0D_t^{\alpha-j-1} f(0) = 0, \quad j = 0, 1, 2, \ldots, n-1 \cdots\cdots (7) $$

then

$$ L\{{}_0D_t^{\alpha} f(0)\} = s^{\alpha}F(s) \cdots\cdots\cdots\cdots\cdots\cdots\cdots (8) $$

Namely, the Laplace transform of fractional order derivative is fractional order Laplace calculator s. Obviously, the Fourier transform of fractional derivative can be obtained by substituting s with $j\omega$ in its Laplace transform just like the classical integer order derivative's. Fractional order calculus is also a generalization of classical integer order calculus in Laplace and Fourier transforms, which implies the future researches of FOC can still make good use of extremely well-developed classical integer order control theory for reference.

3. Integer Order *PID* Control

3.1 Modeling Two-inertia System The most simple model and block diagram of the two-inertia systems are shown in Fig. 1 and Fig. 2, where J_M and J_L are driving side and load side's inertias, K_s is shaft elastic coefficient, ω_M and ω_L are driving side and load side's rotation speeds, T_M and T_L are are the input torque and the disturbance torque respectively.

The open loop transfer function from T_M to ω_M is

$$ G(s) = \frac{s^2 + {\omega_h}^2}{J_M s (s^2 + \omega_o)^2} \cdots\cdots\cdots\cdots\cdots\cdots\cdots (9) $$

As depicted in Fig. 3 the resonance frequency ω_o and the anti-resonance frequency ω_h are

$$ \omega_o = \sqrt{K_s \left(\frac{1}{J_M} + \frac{1}{J_L} \right)}, \ \omega_h = \sqrt{\frac{K_s}{J_L}} \cdots\cdots (10) $$

Parameters of the experimental two-inertia system are shown in Table. 1:

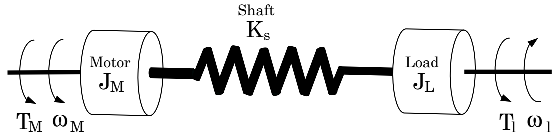

Fig. 1. Two-inertia system model

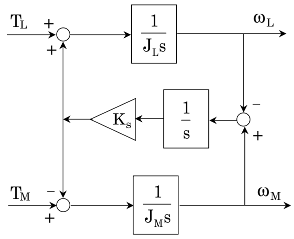

Fig. 2. Block diagram of two-inertia system

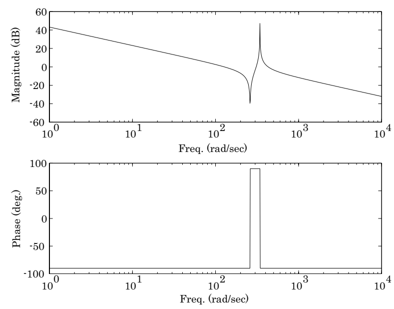

Fig. 3. Bode plot of the two-mass plant

Table 1. Two-inertia System's Parameters

$\mathbf{J_M}$ (Kgm^2)	$\mathbf{J_L}$ (Kgm^2)	$\mathbf{K_s}$ (Nm/rad)
0.004	0.003	198.490

3.2 Design of *PID* Controller In order to smooth the discontinuity of speed command ω_r by the integral controller, a set-point-I *PID* controller is introduced to the speed control of the two-inertia system.

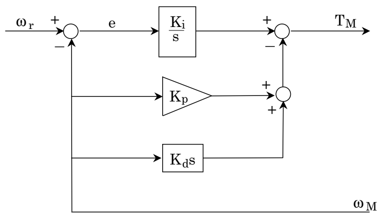

Fig. 4. Set-point-I PID controller

The *PID* controller is designed by Coefficient Diagram Method (CDM) [9], a design method based on the characteristic equations' pole-placement:

$$ K_p = \frac{10\sqrt{2}}{11}\sqrt{J_s K_s}, \ K_i = \frac{4}{11}K_s, \ K_d = \frac{5}{11}J_L - J_M (11) $$

Based on Table. 1 and Equ. (11):

$$ K_p = 0.979, \ K_i = 72.178, \ K_d = -0.003 \cdots\cdots (12) $$

Time responses by simulation show the designed *PID* control system has satisfactory performances (see Fig. 5). However as depicted in Fig. 6, in its frequency response the enough phase margin is not kept in the neighborhood of the critical points, which will lower the integer order PID control system's robustness against non-linearities such as saturation and parameter variation.

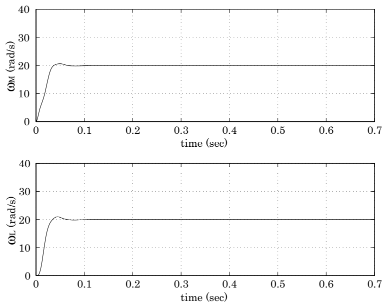

Fig. 5. Time responses by simulation

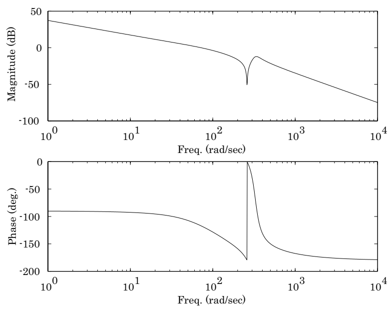

Fig. 6. Bode plot of designed integer order PID control system

4. Fractional Order $PI^\alpha D$ Control

4.1 Frequency Band I^α Controller The most direct way to improve the robustness of the designed PID control system is to adjust I controller's order for giving the control system more phase margin around the critical point. However, it is neither practicable nor desirable to try to make the order be fractional in all frequency range. The frequency-band fractional order controllers are required and practical in real applications. As shown in Equ. (13) a frequency-band I^α controller is propose to substitute classical integer order I controller of the PID controller where the low band frequency $\omega_b = 10rad/sec$ and high band frequency $\omega_h = 1000rad/sec$:

$$\frac{1}{s}\left(\frac{\frac{s}{\omega_b}+1}{\frac{s}{\omega_h}+1}\right)^{1-\alpha} \quad \cdots\cdots\cdots\cdots (13)$$

As depicted in Fig. 7, unlike their integer-order counterparts, by changing order α the $PI^\alpha D$ control system's phase margin can be adjusted directly and continuously. This flexibility in adjusting control systems's frequency responses makes it clear to design robust control systems through the FOC approach.

4.2 Realization Method It is intuitive to approximate fractional order controllers by frequency domain approaches due to their clear geometric interpretations in this domain. In this paper, a broken-line approximation method is introduced to realize frequency-band

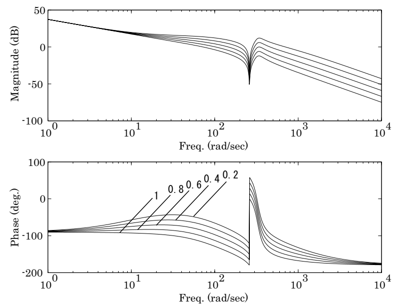

Fig. 7. Bode plots of fractional order $PI^\alpha D$ control systems

fractional order I^α controller. Let

$$D(s) = \left(\frac{\frac{s}{\omega_b}+1}{\frac{s}{\omega_h}+1}\right)^{\alpha} \approx D_N(s) \quad \cdots\cdots\cdots\cdots (14)$$

with

$$D_N(s) = \prod_{i=-N}^{N} \frac{\frac{s}{\omega_i'}+1}{\frac{s}{\omega_i}+1} \cdots\cdots\cdots\cdots (15)$$

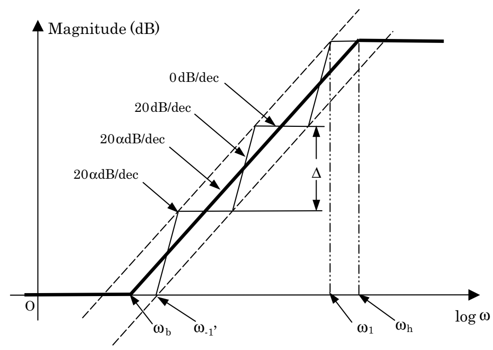

Fig. 8. Broken-line approximation ($N = 1$)

Based on Fig. 8, two recursive factors x and y are introduced to calculate ω_i and ω_i':

$$x = \frac{\omega_i}{\omega_i'},\ y = \frac{\omega_{i+1}'}{\omega_i} \quad \cdots\cdots\cdots\cdots (16)$$

Since

$$\omega_{-N}' = y^{\frac{1}{2}}\omega_b,\ \omega_N = y^{-\frac{1}{2}}\omega_h \quad \cdots\cdots\cdots\cdots (17)$$

Therefore

$$xy = \left(\frac{\omega_h}{\omega_b}\right)^{\frac{1}{2N+1}} \quad \cdots\cdots\cdots\cdots (18)$$

with

$$\omega_i' = (xy)^{i+N}\omega_{-N}',\ \omega_i = x(xy)^{i+N}\omega_{-N}' \quad \cdots\cdots (19)$$

The frequency-band I^α controller has $20\alpha dB/dec$ gain slope, while the integer order factors $s/\omega_i'+1$ have $20dB/dec$ slope. For the same magnitude change Δ:

$$20\alpha = \frac{\Delta}{logx+logy},\ 20 = \frac{\Delta}{logx} \quad \cdots\cdots\cdots\cdots (20)$$

Thus

$$(xy)^{\alpha} = x \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \quad (21)$$

Therefore x and y can be expressed respectively by

$$x = \left(\frac{\omega_h}{\omega_b}\right)^{\frac{\alpha}{2N+1}}, \; y = \left(\frac{\omega_h}{\omega_b}\right)^{\frac{1-\alpha}{2N+1}} \cdots\cdots\cdots\cdots \quad (22)$$

Finally

$$\omega_i^{'} = \left(\frac{\omega_h}{\omega_b}\right)^{\frac{i+N+\frac{1}{2}-\frac{\alpha}{2}}{2N+1}} \omega_b, \; \omega_i = \left(\frac{\omega_h}{\omega_b}\right)^{\frac{i+N+\frac{1}{2}+\frac{\alpha}{2}}{2N+1}} \omega_b \quad (23)$$

Figure. 9 shows the Bode plots of ideal frequency-band $D(s)$ ($\alpha = 0.4,\ \omega_b = 10Hz,\ \omega_h = 1000Hz$) and its 1st-order, 2nd-odes and 3rd-order approximations by the broken-line approximation method. Even taking $N = 2$ can give a satisfactory accuracy in frequency domain.

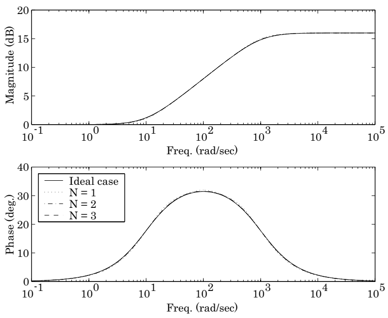

Fig. 9. Bode plots of ideal case and approximations

5. Experimental Results

Experiments of the two-inertia speed control by integer order PID controller and frequency-band $PI^{\alpha}D$ controller are carried out based on the parameters setting in Table. 1 with maximum input torque limitation $T_M = 3.84NM$ and two encoders ($8000pulse/rev$) as rotation speed sensors. The controllers are realized on a digital computer with the sampling time $0.001sec$. In the experiment α is taken as 0.6 to give proper phase margin around the critical point. The parameters K_i, K_p, K_d of PID and $PI^{0.6}D$ controllers are kept as same as the settings in Equ. (12).

For integral order PID controller the step response changes greatly when the input torque T_M saturation occurs, while it can be seen the frequency-band $PI^{0.6}D$ control systems showing better robustness against the saturation non-linearity as depicted in Fig. 10.

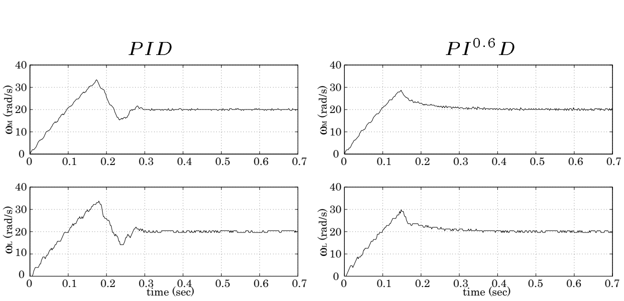

Fig. 10. Step responses with input torque saturation

Fig. 11 gives step responses of the two control systems with different inertia on load side. Compared to the severe change of integer order PID control system's time responses with large overshoot and overswing, the frequency-band $PI^{0.6}D$ control system shows better robustness against inertia variation with much smaller and nearly constant overshoot.

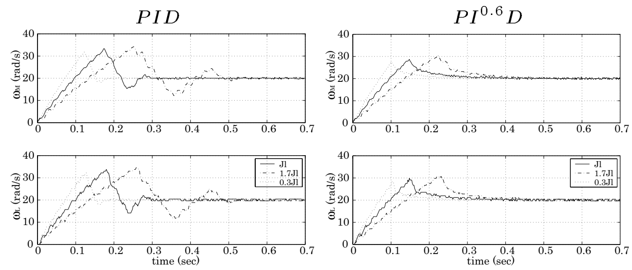

Fig. 11. Step responses with load inertia variation

6. Conclusions

In this paper, a frequency-band fractional order $PI^{\alpha}D$ controller is proposed for the speed control of two-inertia system with input torque saturation and load side inertia variation. An intuitive broken-line approximate realization method of the frequency-band I^{α} controller is also introduced which has a satisfactory accuracy in frequency domain. The experimental results show the robustness of proposed fractional order $PI^{\alpha}D$ control system. By changing the fractional order, control system's robustness can be directly improved which means clearer design and less tuning efforts in real industrial applications. Applying FOC to motion control is still in a research stage, but its superior robustness against parameter variation and non-linearities shown in the experimental results highlights the promising aspects while future exploration of the applications to more complex cases is needed.

References

(1) Oldham, K. B., and Spanier, J., 1974, *The Fractional Calculus*, Academic Press, New York and London.

(2) Tustin, A., and others, 1958, "The design of Systems for Automatic Control of the Position of Massive Objects", *Proc., IEE*, **105-C**(1), pp. 1-57

(3) Axtell, M., and Bise, M. E., 1990, "Fractional Calculus Applications in Control Systems", *Proc., the IEEE 1990 Nat. Aerospace and Electronics Conference*, New York, USA, pp. 563-566.

(4) I. Podlubny, Fractional Differential Equations, Vol. 198, Mathematics in Science and Engineering, New York Tokyo, Academic Press, 1999

(5) Vinagre, B. M., Feliú, V., and Feliú, J. J., 1998, "Frequency Domain Identification of a Flexible Structure with Piezoelectric Actuators Using Irrational Transfer Function", *Proc., the 37th IEEE Conference on Decision & Control*, Tampa, Florida,USA,pp. 1278-1280.

(6) Tenreiro Machado, J. A., 1997, " Theory Analysis and Design of Fractional-Order Digital Control Systems", Journal Systems Analysis-Modeling-Simulation, **27**, pp. 107-122.

(7) Oustaloup, A., Sabatier, J., and Moreau, X., 1998, "From Fractal Robustness to the CRONE Approach", *Proc., ESAIM*, Vol. 5, pp. 177-192.

(8) Petras, I., and Vinagre,B. M., 2001, "Practical Application of Digital Fractional-order Controller to Temperature Control", Acta Montanistica Slovaca, Kosice , 2001

(9) Hori,Y., 1995, "Control of 2-Inertia System only by a PID Controller", T. IEE Japan, **112-D**(5), pp. 499-500