Fractional Order Control and Its Application of $\mathrm{PI}^{\alpha}\mathrm{D}$ Controller for Robust Two-inertia Speed Control

Chengbin Ma
Institute of Industrial Science
the University of Tokyo
4-6-1 Komaba, Meguro-ku, Tokyo, Japan
Email: ma@horilab.iis.u-tokyo.ac.jp

Yoichi Hori
Institute of Industrial Science
the University of Tokyo,
4-6-1 Komaba, Meguro-ku, Tokyo, Japan
Email: hori@iis.u-tokyo.ac.jp

***Abstract*— This paper deals with the speed control of two-inertia system by fractional order $PI^{\alpha}D$ controller which means the order of I controller will not only be integer but also can be any real number. The significance of fractional order control is that it is a generalization and "interpolation" of the classical integer order control theory, which can achieve more adequate modeling and clear-cut design of robust control system. However, most of fractional order control researches were originated and concentrated on the control of chemical processes, while in motion control the research is still in a primitive stage. In this paper, a frequency-band fractional order $PI^{\alpha}D$ controller is proposed to speed control of the two-inertia system, which is a basic control problem in motion control. A frequency-band broken-line approximation method is introduced to realize the designed fractional order $PI^{\alpha}D$controller that has a satisfactory accuracy in frequency domain. The better robustness performances of the $PI^{\alpha}D$ control system against saturation non-linearity and load inertia variation are shown by the comparison of fractional order $PI^{\alpha}D$ control's experimental time responses with integer order PID control's. The superior robustness and clear-cut control design highlight the promising aspects of applying fractional order control in motion control.**

keywords - fractional order control; two-inertia system, speed control; robustness

I. Introduction

A. History Review

Fractional Order Control (FOC) means controlled systems and/or controllers described by fractional order differential equations. Expanding calculus to fractional orders is by no means new and actually had a firm and long standing theoretical foundation. Leibniz mentioned it in a letter to L'Hospital over three hundred years ago (1695). The earliest more or less systematic studies seem to have been made in the beginning and middle of the 19th century by Liouville (1832), Holmgren (1864) and Riemann (1953), although Eular, Lagrange, and others made contribution even earlier [1] [2].

As one of fractional order calculus's applications in control engineering, FOC was introduced by Tustin for the position control of massive objects half a century ago, where actuator saturation requires sufficient phase margin around and below the critical point [3]. For such kind of $1/s^{\alpha}$ system, taking fractional order α $(1 < \alpha < 2)$ will give a desirable tradeoff between control system's stability and robustness against saturation non-linearity. Some other pioneering works were also carried out around 60's by Manabe [4]. However FOC was not widely incorporated into control engineering mainly due to the unfamiliar idea of taking fractional order, so few physical applications and limited computational power available at that time [5].

B. Present Situation

In last few decades, researchers found that fractional order differential equations could model various materials more adequately than integer order ones and provide an excellent tool for describing dynamic processes [1] [2] [6]. The fractional order models need fractional order controllers for more effective control of dynamic systems [7]. This necessity motivated renewed interest in various applications of FOC [8] [9] [10]. And with the rapid development of computer performances, modeling and realization of FOC systems also became possible and much easier than before.

The researches on FOC are mainly centered in European universities at present. The CRONE (Non-integer order robust control in France) team in France is leaded by Alain Oustaloup and Patrick Lanusse from Bordeaux University, France. Their practices include applying FOC into car suspension control, flexible transmission, hydraulic actuator etc. Denis Matignon, a researcher from ENST, Signal Dept. & CNRS, URA, France, contributed to the theoretical aspects of FOC concept, such as stability, controllability, and observability of the fractional order systems. Slovak researchers, Ivo Petras and Igor Podlubny from the Technical University of Kosice, are well-known for their efforts in modeling, realization and implementation of fractional order systems. J. A. Tenreiro Machado and Yangquan Chen, from Polytechnic Institute of Porto, Portugal, and Utah State University, Logan, are playing important roles in developing the implementation methods for fractional order controllers and applying FOC in robotics control, disturbance observer, etc.

Fractional differentiation's applications in automatic control is now an important issue for the international scientific community. The First Symposium on Fractional Derivatives and Their Applications (FDTA) of the 19th Biennial Conference on Mechanical Vibration and Noise was held from September 2 to September 6, 2003 in Chicago, IL, USA [11]. This conference was part of the ASME 2003 Design Technical Conferences. 29 papers concerning FDTA in Automatic Control, Automatic Control and System, Robotics and Dynamic Systems, Analysis Tools and Numerical Methods, Modeling, Visco-elasticity and Thermal Systems were presented in the symposium.

A sub-committee called "Fractional Dynamic System" under ASME "Multi-body Systems and Nonlinear Dynamics" committee was formed during the symposium. And first IFAC Workshop on Fractional Differentiation and its Applications will be held in this year's summer, July 19-21, in Bordeaux, France [12]. The following areas will be covered by the workshop: Representation tools, analysis tools, synthesis tools, simulation tools, modeling, identification, observation, control, vibration insulation, filtering, pattern recognition, edge detection. Besides the presentation of theoretical works and applications, this workshop can also give rise to benchmark, testing bench and software presentations.

The article is organized as follows: in section II, mathematical aspects of fractional order control are mentioned; in section III, a integer order PID controller is designed for the speed control; in section IV, a frequency band $PI^{\alpha}D$ controller is proposed and its broken-line realization method is also introduced; in Section V, Experimental results are presented to show the robustness of proposed fractional order $PI^{\alpha}D$ controllers. Finally, in section V, conclusions are drawn.

II. Mathematical Aspects

A. Mathematical Definitions

The mathematical definition of fractional derivatives and integrals has been the subject of several different approaches[1][2]. The most frequently encountered definition is called Riemann-Liouville definition, in which the fractional order integrals are defined as

$$ {}_{t_0}D_t^{-\alpha} = \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t-\xi)^{\alpha-1} f(\xi) d(\xi) \qquad (1) $$

while the definition of fractional order derivatives is

$$ {}_{t_0}D_t^{\alpha} = \frac{d^n}{dt^n} \left[{}_{t_0}D_t^{-(n-\alpha)} \right] \qquad (2) $$

where

$$ \Gamma(x) \equiv \int_0^{\infty} y^{x-1} e^{-y} dy \qquad (3) $$

is the Gamma function, t_0 and t are limits and α ($\alpha > 0$ and $\alpha \in R$) is the order of the operation. n is an integer that satisfies $(n-1) < \alpha < n$.

The other approach is Grünwald-Letnikov definition:

$$ {}_{t_0}D_t^{\alpha} = \lim_{\substack{h \to 0 \\ nh = t - t_0}} h^{-\alpha} \sum_{r=0}^{n} (-1)^{\alpha} f(t - rh) \qquad (4) $$

Where the binomial coefficients $(r > 0)$

$$ \left(\begin{array}{c} \alpha \\ 0 \end{array} \right) = 1, \; \left(\begin{array}{c} \alpha \\ r \end{array} \right) = \frac{\alpha(\alpha-1)\dots(\alpha-r+1)}{r!} \qquad (5) $$

B. Laplace and Fourier Transforms

The Laplace transform of the Riemann-Liouville fractional order derivative with order $\alpha > 0$ [1] [2] is

$$ L\left\{ {}_0D_t^{\alpha} \right\} = s^{\alpha} F(s) - \sum_{j=0}^{n-1} s^j \left[{}_0D_t^{\alpha-j-1} f(0) \right] \qquad (6) $$

where $(n-1) \leq \alpha < n$. If

$$ {}_0D_t^{\alpha-j-1} f(0) = 0, \quad j = 0, 1, 2, \dots, n-1 \qquad (7) $$

then

$$ L\left\{ {}_0D_t^{\alpha} f(0) \right\} = s^{\alpha} F(s) \qquad (8) $$

Namely, the Laplace transform of fractional order derivative is fractional order Laplace operator s. The Fourier transform of fractional derivative can be obtained by substituting s with $j\omega$ in its Laplace transform just like the classical integer order derivative's. Fractional order calculus is also a generalization of classical integer order calculus in Laplace and Fourier transforms.

Obviously, the fractional order system's frequency responses can be exactly known. The researches of FOC can still make good use of extremely well-developed classical integer order control theory for reference, especially in frequency domain.

III. Integer Order PID Control

A. Two-inertia Modeling

The testing bench of torsional system is depicted in Fig. 1, which is a typical oscillatory system. Two flywheels are connected with a long torsional shaft; while driving force is transmitted from driving servomotor to the shaft by gears with gear ratio 1:2. Some system parameters are adjustable, such as gear inertia, load inertia, shaft's elastic coefficient and gears' backlash angle. The encoders and tacho-generators are used as position and rotation speed sensors.

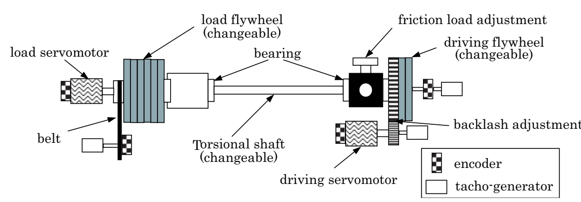


Figure 1. Experimental setup of torsional system

The most simple model for the torsional system is the two-inertia model, as shown in Fig. 2 and Fig. 3. J_M and J_L are the inertias of driving side (including motor, gear and driving flywheels) and load side, K_S shaft elastic coefficient, ω_M and ω_L motor and load rotation speeds, T_M input torque and T_L disturbance torque. In this paper, the backlash angle is set to be zero. Frictions between components are neglected due to their small values.

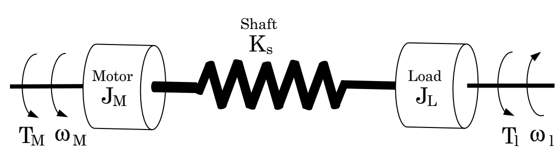


Figure 2. Two-inertia system model

The open loop transfer function between T_M to ω_M is

$$ G(s) = \frac{s^2 + {\omega_h}^2}{J_M s (s^2 + \omega_o)^2} \qquad (9) $$

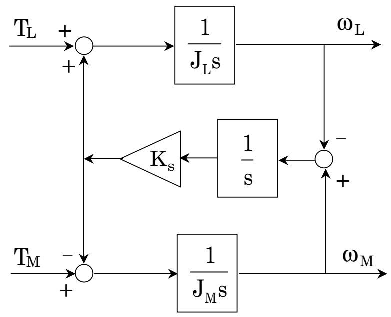


Figure 3. Block diagram of two-inertia system

As shown in Fig. 4 the resonance frequency ω_o and the anti-resonance frequency ω_h are

$$\omega_o = \sqrt{K_s\left(\frac{1}{J_M}+\frac{1}{J_L}\right)},\ \omega_h = \sqrt{\frac{K_s}{J_L}} \tag{10}$$

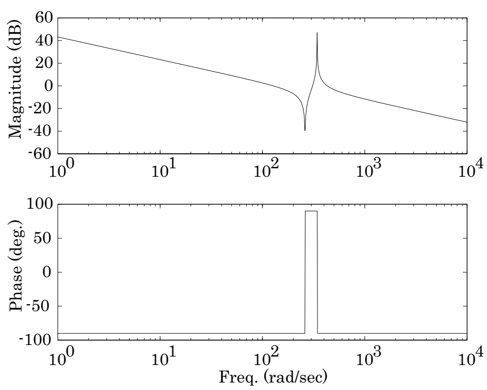


Figure 4. Bode plot of the two-mass plant

Parameters of the experimental two-inertia system are shown in Table. I:

TABLE I. TWO-INERTIA SYSTEM'S PARAMETERS

$\mathbf{J_M}$ (Kgm^2)	$\mathbf{J_L}$ (Kgm^2)	$\mathbf{K_s}$ (Nm/rad)
0.004	0.003	198.490

B. Design of PID Controller

A set-point-I PID controller is introduced to speed control of the two-inertia system:
where the PID controller's parameters are

$$K_p = \frac{10\sqrt{2}}{11}\sqrt{J_s K_s},\ K_i = \frac{4}{11}K_s,\ K_d = \frac{5}{11}J_L - J_M \tag{11}$$

which is designed by standard form of Coefficient Diagram Method [13] [14], a design method based on the characteristic equations' pole-placement. Based on Table. I and Equ. (11):

$$K_p = 0.979,\ K_i = 72.178,\ K_d = -0.003 \tag{12}$$

Time responses by simulation show the designed PID control system has satisfactory performances (see Fig. 6).

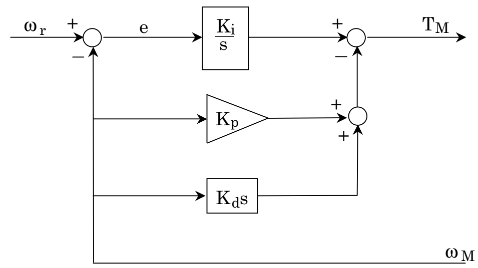


Figure 5. Set-point-I PID controller

While in its frequency response, the enough phase margin is not kept in the neighborhood of the critical points, which will lower the integer order PID control system's robustness when non-linearities such as saturation and parameter variations occur, as shown in Fig. 7.

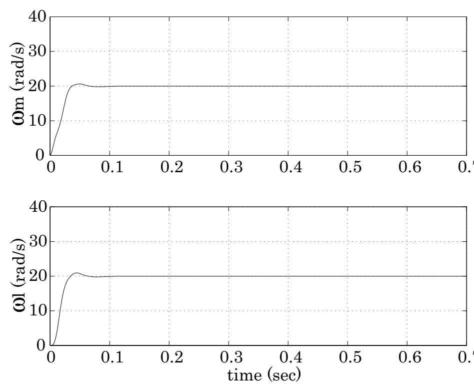


Figure 6. Time responses by simulation

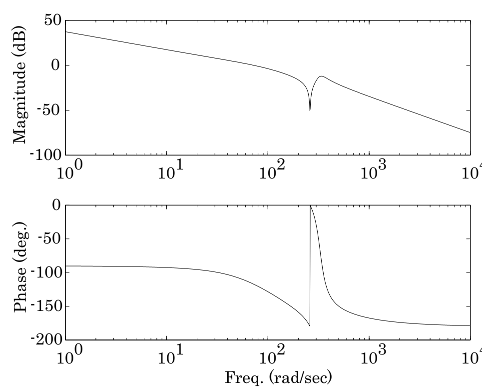


Figure 7. Bode plot of designed integer order PID control system

IV. FRACTIONAL ORDER $PI^{\alpha}D$ CONTROL

A. Frequency Band I^{α} Controller

The most clear-cut way to enhance the robustness of designed PID control system is to adjust I controller's order for giving the control system more phase margin around the critical point. However, it is neither practicable nor desirable to try to make the order be fractional in all frequency range. The frequency-band fractional order controllers are required and practical in real applications. As shown in Equ. (13) a frequency-band I^{α} controller is propose to substitute classical integer order I controller where the low band frequency $\omega_b = 10rad/sec$ and high band frequency $\omega_h = 1000rad/sec$:

$$\frac{1}{s}\left(\frac{\frac{s}{\omega_b}+1}{\frac{s}{\omega_h}+1}\right)^{1-\alpha} \tag{13}$$

By changing the order of α, the phase margin of proposed fractional order $PI^{\alpha}D$ control system can be adjusted directly to any desired amount (see Fig. 8). As shown in Fig. 9 and Fig. 10, when uncertainties such as saturation (gain variation) and load inertia variation occur, a enough phase margin can be easily kept by choosing proper fractional order α.

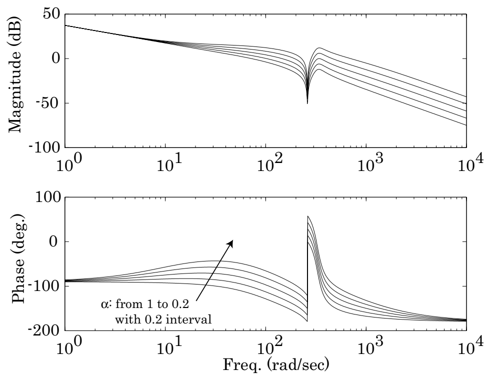


Figure 8. Bode plots of fractional order $PI^{\alpha}D$ control systems

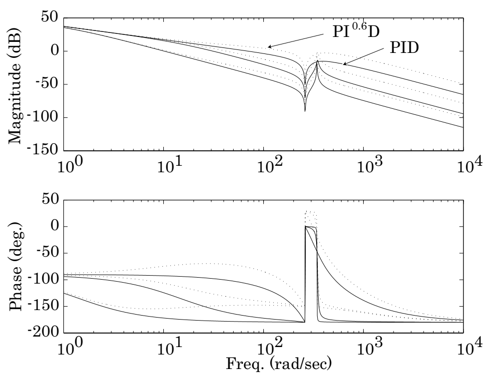


Figure 9. Bode plots with gain variation

B. Realization Method

Fractional order systems have an infinite dimension while integer order systems are finite dimensional. Proper approximation by finite difference equation is needed. There are various way to realize designed fractional order controllers, such as Short Memory Principle, Sampling Time Scaling, Tustin Taylor Expansion and Lagrange function interpolation, etc [15].

It is intuitive to approximate fractional order controllers by frequency domain approach due to their clear geometric interpretation in this domain. A broken-line approximation

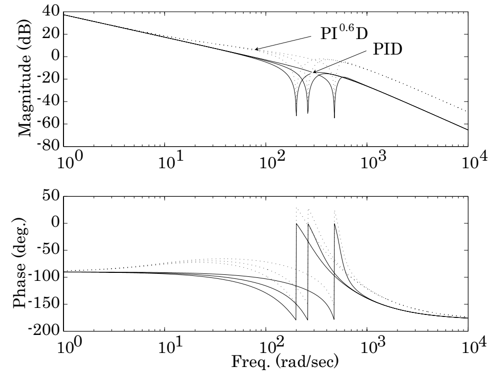


Figure 10. Bode plots with J_L variation

method is introduced to realize frequency-band fractional order controllers. Let

$$\left(\frac{\frac{s}{\omega_h}+1}{\frac{s}{\omega_b}+1}\right)^{\alpha} \approx \prod_{i=0}^{N-1} \frac{\frac{s}{\omega_i'}+1}{\frac{s}{\omega_i}+1} \tag{14}$$

Based on Fig. 11, two recursive factors ζ and η are

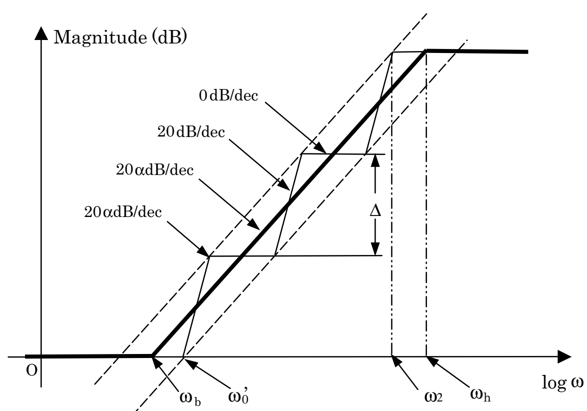


Figure 11. An example of broken-line approximation ($N = 3$)

introduced to calculate ω_i and ω_i':

$$\zeta = \frac{\omega_i}{\omega_i'},\ \eta = \frac{\omega_{i+1}'}{\omega_i} \tag{15}$$

Since

$$\omega_0' = \eta^{\frac{1}{2}}\omega_b,\ \omega_{N-1} = \eta^{-\frac{1}{2}}\omega_h \tag{16}$$

Therefore

$$\zeta\eta = \left(\frac{\omega_h}{\omega_b}\right)^{\frac{1}{N}} \tag{17}$$

with

$$\omega_i' = (\zeta\eta)^i\omega_0',\ \omega_i = \zeta(\zeta\eta)^i\omega_0' \tag{18}$$

The frequency-band fractional order controller has $20\alpha dB/dec$ gain slope, while the integer order factors $\frac{s}{\omega_i'}+1$ have $20dB/dec$ slope. For the same magnitude change Δ:

$$20\alpha = \frac{\Delta}{log\zeta + log\eta},\ 20 = \frac{\Delta}{log\zeta} \tag{19}$$

Thus

$$(\zeta\eta)^{\alpha} = \zeta \tag{20}$$

Therefore ζ and η can be expressed respectively by

$$\zeta = \left(\frac{\omega_h}{\omega_b}\right)^{\frac{\alpha}{N}},\ \eta = \left(\frac{\omega_h}{\omega_b}\right)^{\frac{1-\alpha}{N}} \tag{21}$$

Finally

$$\omega_i^{'} = \left(\frac{\omega_h}{\omega_b}\right)^{\frac{i+\frac{1}{2}-\frac{\alpha}{2}}{N}}\omega_b,\ \omega_i = \left(\frac{\omega_h}{\omega_b}\right)^{\frac{i+\frac{1}{2}+\frac{\alpha}{2}}{N}}\omega_b \tag{22}$$

Figure. 12 shows the Bode plots of ideal frequency-band case ($\alpha = 0.4,\ \omega_b = 200Hz,\ \omega_h = 1000Hz$) and its 1st-order, 2nd-order and 3rd-order approximations by broken-line approximation method. Even taking $N = 3$ can give a satisfactory accuracy in frequency domain.

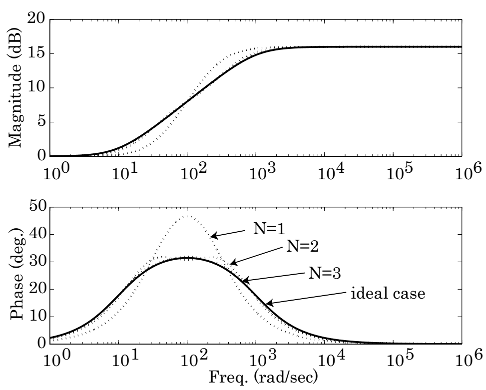


Figure 12. Bode plots of ideal case, 1st, 2nd and 3rd-order approximations

The transfer function of the 2nd-order approximation $C_f(s)$ for the frequency-band 0.4 order controller is

$$\frac{6.3096(s+15.85)(s+73.56)(s+341.5)}{(s+29.29)(s+135.9)(s+631)} \tag{23}$$

The controller can be discretized by bilinear approximation. When the sampling time is $0.001sec$, the discrete controller $Z\{C_f(s)\}$ is

$$\frac{5.415z^3 - 14.2z^2 + 12.29z - 3.508}{z^3 - 2.364z^2 + 1.807z - 0.441} \tag{24}$$

V. Experimental Results

As shown in Fig. 13, the experimental torsional system is controlled by a PC with 1.6GHz Pentium IV CPU and 512M memory. Control programs are written in RTLinux C threads which can be executed with strict timing requirement of control sampling time. A 12-bit AD/DA multi-functional board is used whose conversion time per channel is $10\mu sec$.

Experiments are carried out with sampling time T=0.001sec and 3rd-order broken-line approximation ($N = 2$). Two encoders ($8000 pulse/rev$) are used as rotation speed sensors with coarse quantization $\pm 0.785 rad/sec$.

Experiments of two-inertia speed control by integer order PID controller and frequency-band $PI^{\alpha}D$ controller are carried out based on the parameters setting in Table. I with maximum output torque limitation $\pm 3.84NM$. The

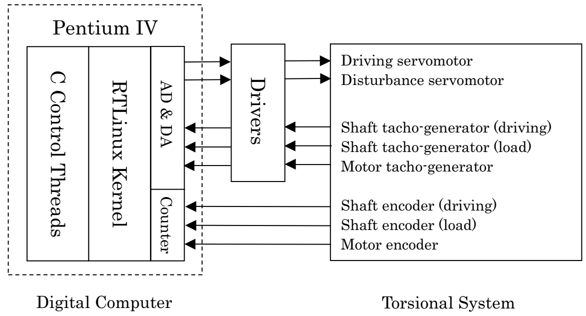


Figure 13. Digital control system of the experimental setup

parameters K_i, K_p, K_d of PID and $PI^{\alpha}D$ controllers are kept as same as the settings in Equ. 12.

As shown in Fig. 14 and 15, letting I controller's order be fractional can affect control system's time response greatly. It can be seen the frequency-band $PI^{\alpha}D$ systems show better robustness to saturation non-linearity with smaller overshoots.

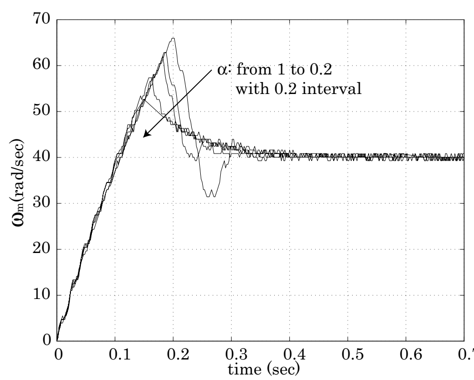


Figure 14. Step responses of driving motor with input torque saturation

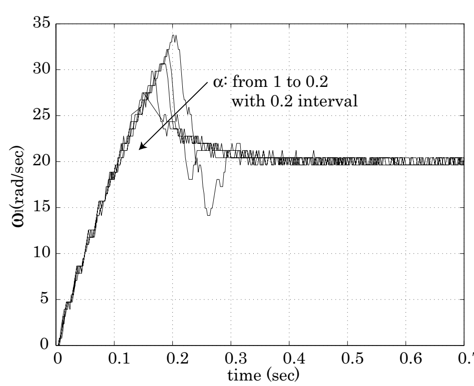
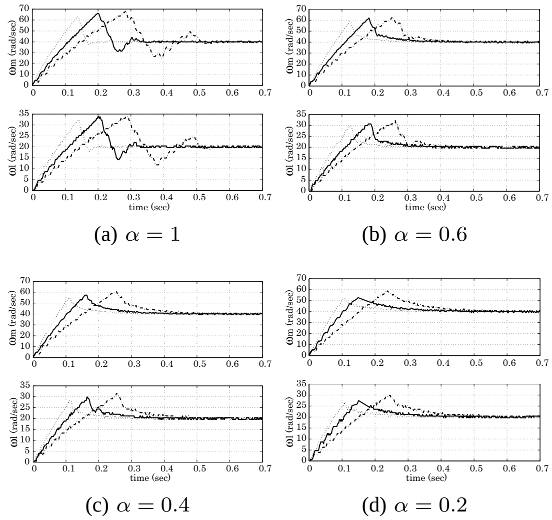


Figure 15. Step responses of load side with input torque saturation

Fig. 16 gives the step responses of the $PI^{\alpha}D$ control systems with different inertia on load side. Compared to the severe change of integer order PID control system's time responses with large overshoot and overswing, the frequency-band $PI^{\alpha}D$ control systems show better robustness against load inertia variation.



(a) $\alpha = 1$ (b) $\alpha = 0.6$

(c) $\alpha = 0.4$ (d) $\alpha = 0.2$

Figure 16. Step responses of $PI^{\alpha}D$ system with load inertia variation. (solid line: nominal case; dotted line: $0.3J_l$; broken line: $1.7J_l$)

VI. Conclusions

In this paper, a frequency-band fractional order $PI^{\alpha}D$ controller is proposed for the speed control of two-inertia system with input torque saturation and load side inertia variation. An intuitive broken-line approximate realization method of frequency-band controllers is also introduced which has a satisfactory accuracy in frequency domain. The experimental results show the robustness of proposed fractional order $PI^{\alpha}D$ control system. By changing the fractional order, the robustness of control system can be directly improved which means easier design process and less tuning efforts in real industrial applications.

It can be seen that even a relative high-order controller is actually introduced, the design process is really clear-cut. Only two control parameters, the fractional order α and approximation order N need to be decided during the designing. The whole design process shows a good prospect. Generally, there are three main advantages for introducing FOC to control design:

- More adequate modeling of dynamic systems
- More clear-cut robust control design
- Reasonable implementation by approximation

By introducing FOC approach, control system's phase and gain responses can be easily offset to any desired amount. Design process and experimental results demonstrate a clear-cut and effective robust control design is possible based on FOC approach. On the contrary to FOC control design, the implementation of fractional order controllers is not such straightforward. Some proper approximations are needed. However, as verified in experimental results, implementation issue is actually not problematic.

FOC should not be an independent concept of the well-developed IOC. Knowledge and design methods developed in IOC can still be made full use of in FOC research, as demonstrated in this paper. It is interesting to notice that even the theoretical analysis and design are based FOC approach, the implementation of fractional order controllers are certainly integer order controllers. Therefore, FOC should not be thought as a novel and conceptually difficult idea, but actually a natural and more effective control design tool. By FOC, control system's responses can be designed with much more flexibility. The integer order controller's structure and parameters can be decided by only one parameter, the fractional order. This enlarged flexibility will provide more possibility to find excellent solutions with less design effort. Some well-designed IOC system might be looked as a good approximation of FOC system. If this hypothesis can be established, it will further verify FOC's advantages in control field.

Fractional order control in motion control is still in a research stage, but its superior robustness against parameter variation and non-linearities shown in the experimental results highlights the promising aspects; while future exploration of the applications to more complex cases is needed. Systematic and clear-cut FOC control design methods are also important for promoting the application of the novel FOC concept.

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