The Application of Fractional Order Control to Backlash Vibration Suppression

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***Abstract*— This paper applies fractional order PID^k controller to torsional system's backlash vibration suppression control, in which the order k of D controller can not only be integer but also be any real number. In order to improve control system's robustness against backlash non-linearity, several methods have been proposed. However their design processes are very complicated. In this paper, a clear and straightforward design is achieved by adjusting the D controller's order k directly. An approximation method based on sampling time scaling property is also introduced to realize the discrete D^k controller. Design process and experimental results demonstrate straightforward robust control design through the novel Fractional Order Control (FOC) approach, the PID^k control system's robustness against backlash non-linearity and good approximation of the realization method.**

I. Introduction

The concept of Fractional Order Control (FOC) means controlled systems and/or controllers are described by fractional order differential equations. Expanding derivatives and integrals to fractional orders has a firm and long standing theoretical foundation. $Leibniz$ mentioned this concept in a letter to $L'Hospital$ over three hundred years ago in 1695 and the earliest more or less systematic studies have been made in the beginning and middle of the 19th century by $Liouville$, $Riemann$ and $Holmgren$ [1] [2]. As to its application in control engineering, FOC was introduced by $Tustin$ for the position control of massive objects half a century ago, where actuator saturation requires sufficient phase margin around and below the critical point [3]. Some pioneering works were also done in 60's [4]. However the FOC concept was not widely incorporated into control engineering mainly due to the conceptually difficult idea of taking fractional order, the existence of so few physical applications and the limited computational power available at that time [5].

In the last few decades, researchers pointed out that fractional order differential equations could model various materials more adequately than integer order ones and provide an excellent tool for describing dynamic processes [1] [2] [6]. The fractional order models need fractional order controllers for more effective control of the dynamic systems [7]. At the same time, letting control order be fractional can adjust control system's frequency response directly and continuously. This great flexibility makes it possible to design more robust control system. The superiorities of FOC in modeling and control design motivated renewed interest in various applications of FOC [8] [9] [10]. With the rapid development of computer performances, modeling and realization of the FOC systems also became possible and much easier than before.

Just like the other new control theories, "find the problem" is important for its development in engineering. The authors believe the design of FOC systems should be straightforward and there is no reason that we don't make full use of extremely well developed classical integer order control theories. Based on these basic considerations, in this paper, the authors proposed a fractional PID^k controller, a revised version of PID controller, to achieve a straightforward design of torsional system's robust speed control against gear backlash non-linearity. And the fractional order PID^k controller is discretized by using sampling time scaling property which was proposed by the authors [11].

II. Experimental Torsional System

The experimental setup of torsional system is depicted in Fig. 1. A torsional shaft connects two flywheels while driving force is transmitted from driving servomotor to the shaft by gears with gear ratio 1:2. Some system parameters are changeable, such as gear inertia, load inertia, shaft's elastic coefficient and gears' backlash angle. The encoders and tacho-generators are used as position and rotation speed sensors.

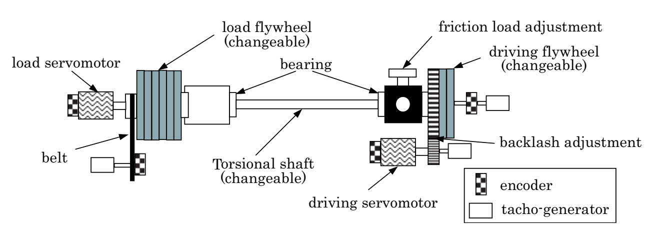

Fig. 1. Experimental setup of torsional system

The simplest model of the torsional system with gear backlash is the three-inertia model depicted in Fig. 2 and Fig. 3, where J_m, J_g and J_l are driving motor, gear (driving flywheel) and load's inertias, K_s shaft elastic coefficient,

ω_m and ω_l motor and load rotation speeds, T_m input torque and T_l disturbance torque. In the modeling, the gear backlash is simplified as a deadzone factor with backlash angle band [-δ,+δ] and elastic coefficient K_g. Frictions between components are neglected due to their small values.

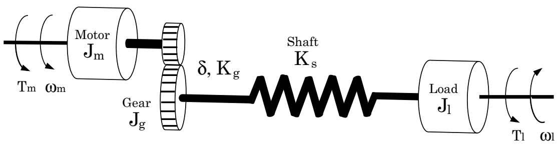

Fig. 2. Torsional system's three-inertia model

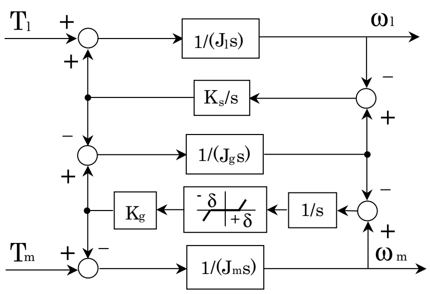

Fig. 3. Block diagram of the three-inertia model

The open-loop transfer function from T_m to ω_m is

$$P_{3m}(s) = \frac{(s^2+\omega_{h1}^2)(s^2+\omega_{h2}^2)}{J_m s(s^2+\omega_{o1}^2)(s^2+\omega_{o2}^2)} \tag{1}$$

where ω_{o1} and ω_{o2} are resonance frequencies while ω_{h1} and ω_{h2} are anti-resonance frequencies. ω_{o1} and ω_{h1} correspond to torsion vibration mode, while ω_{o2} and ω_{h2} correspond to gear backlash vibration mode (see Fig. 4).

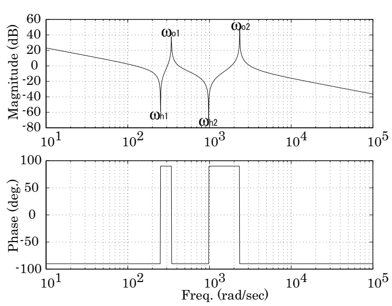

Fig. 4. Bode plot of the three-inertia model

III. Fractional Order PID^k Control

A. Classical PID Control

In PID speed control design, two-inertia model is commonly used in which driving motor inertia J_m and gear inertia J_g are simplified to single inertia $J_{mg}(= J_m + J_g)$ (see Fig. 5).

In order to smooth the discontinuity of speed command ω_r by integral controller, a set-point-I PID controller

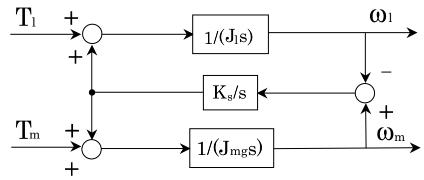

Fig. 5. Block diagram of the two-inertia model

is proposed for the torsional system's speed control (see Fig. 6).

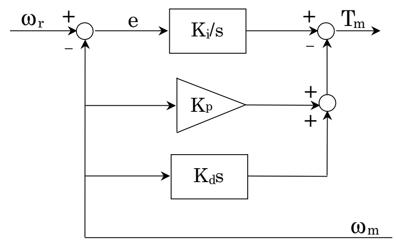

Fig. 6. Set-point-I PID controller

Based on the two-inertia model, the PID controller's parameters are designed by using the standard form of Coefficient Diagram Method $(\gamma_1 = 2.5,\ \gamma_2 = \gamma_3 = 2)$ [12]:

$$K_p = \frac{10\sqrt{2}}{11}\sqrt{J_l k_s}, K_i = \frac{4}{11}K_s, K_d = \frac{5}{11}J_l - J_m \tag{2}$$

Simulation results show the integer order PID control system has a good performance for suppressing torsion vibration (see Fig. 7).

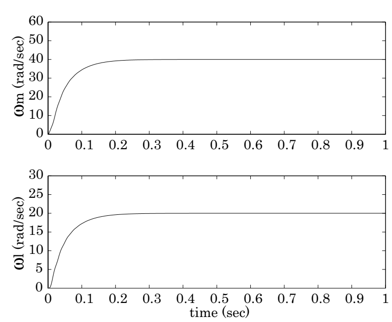

Fig. 7. Time responses of the integer order PID two-inertia system by simulation

For three-inertia plant $P_{3m}(s)$, the close-loop transfer function of integer order PID control system from ω_r to ω_m is

$$G_{close}(s) = \frac{C_I(s)P_{3m}(s)}{1 + C_I(s)P_{3m}(s) + C_{PD}(s)P_{3m}(s)} \tag{3}$$

where $C_I(s)$ is I controller and $C_{PD}(s)$ is the parallel of P and D controllers in minor loop; therefore $G_{close}(s)$ is stable if and only if $G_l = C_I(s)P_{3m}(s) + C_{PD}(s)P_{3m}(s)$ has positive gain margin and phase margin. However as depicted in Fig. 8 the gain margin of $G_l(s)$ is negative. With the existence of gear backlash the designed integer order

PID control system will be unstable and lead to backlash vibration.

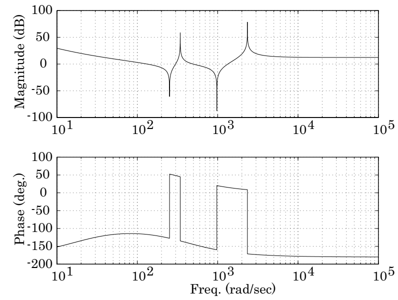

Fig. 8. Bode plot of $G_l(s)$ in PID control

B. Novel PID^k control

In order to improve the robustness against backlash non-linearity, several methods have been proposed, such as redesigning the PID controller, QFT or using H_∞ robust control method [13] [14]. A FOC approach was also practiced by CRONE team [15]. However, their design processes are very complicated. In this paper, a novel fractional order PID^k controller is proposed to achieve a straightforward design of robust control system against gear backlash non-linearity. Instead of solving high order equations, by changing D^k controller's fractional order k, the frequency response of $G_l(s)$ can be directly adjusted (see Fig. 9).

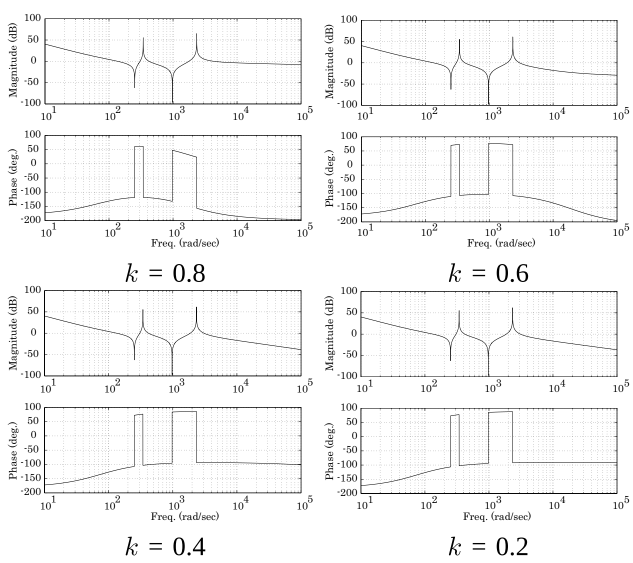

Fig. 9. Bode plots of $G_l(s)$ in PID^k control

As depicted in Fig. 10, letting k be fractional order can improve PID^k control system's gain margin continuously. When $k < 0.84$ the PID^k control system will be stable; therefore, with proper selected fractional order k, the backlash vibration can be suppressed. At the same time, for better backlash vibration suppression performance higher D^k controller's order is more preferable. As shown in open-loop gain plots of 0.8, 0.6, 0.4 and 0.2 order PID^k control systems (see Fig. 11), higher the D controller's order is taken lower the gain near gear backlash vibration mode is. There is a trade-off between robustness and vibration suppression performance in fractional order PID^k approach.

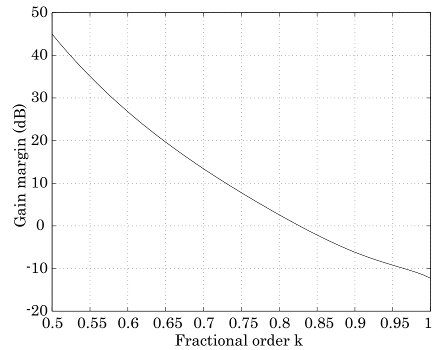

Fig. 10. Gain margin versus fractional order k

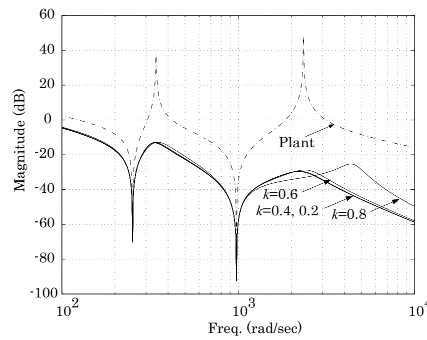

Fig. 11. Gain plots of the PID^k control systems and three-inertia plant

IV. Realization Method

A. Sampling time scaling

From the Riemann-Liouville definition [1] [2], fractional order integral with order between 0 and 1 is

$$ {}_0I_t^{\alpha} f(t) = \int_0^t f(\tau) dg_t(\tau),\ 0 < \alpha < 1 \tag{4}$$

where

$$ g_t(\tau) = \frac{1}{\Gamma(1+\alpha)}[t^{\alpha} - (t-\tau)^{\alpha}] \tag{5}$$

Let $t := nT$, where T is the sampling time and n is the step currently under execution, then

$$ g_{nT}(kT) = \frac{n^{\alpha} - (n-k)^{\alpha}}{\Gamma(1+\alpha)} T^{\alpha},\ k = 1, ..., n \tag{6}$$

Therefore, by sharing the same view of discrete integer order integration rules, the "real" sampling time T of the

kth step is

$$\begin{aligned} T_n(k) &= \Delta g_{nT}(kT) \\ &= g_{nT}(kT) - g_{nT}[(k-1)T] \\ &= \frac{(n-k+1)^\alpha - (n-k)^\alpha}{\Gamma(1+\alpha)} T^\alpha \end{aligned} \tag{7}$$

Thus

$$\begin{aligned} T_n(n) &= \frac{1^\alpha - 0^\alpha}{\Gamma(1+\alpha)} T^\alpha \\ T_n(n-1) &= \frac{2^\alpha - 1^\alpha}{\Gamma(1+\alpha)} T^\alpha \\ &\cdots \\ T_n(1) &= \frac{n^\alpha - (n-1)^\alpha}{\Gamma(1+\alpha)} T^\alpha \end{aligned} \tag{8}$$

Finally, based on the trapezoidal integration rule

$${}_0I_{nT}^\alpha \approx \sum_{k=1}^{n} \frac{f(kT) + f[(k-1)T]}{2} T_n(k) \tag{9}$$

and if $t \to 0$, then

$${}_0I_{nT}^\alpha = \sum_{k=1}^{n} \frac{f(kT) + f[(k-1)T]}{2} T_n(k) \tag{10}$$

From (8), we can see that the interpretation of discrete fractional order integrals is the “deformation” of their integer order counterparts by internal sampling time scaling (see Fig. 12).

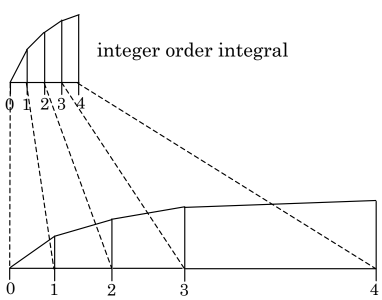

Fig. 12. Fractional order integral’s sampling time scaling

Similarly, discrete fractional order derivatives with order between 0 and 1 is

$$\begin{aligned} {}_0D_t^\alpha f(t) &= \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t \frac{f(\tau)}{(t-\tau)^\alpha} d\tau \\ &= \frac{d[\int_0^t f(\tau) dg_t^{'}(\tau)]}{dt}, \quad 0 < \alpha < 1 \end{aligned} \tag{11}$$

where

$$g_t^{'}(\tau) = \frac{1}{\Gamma(2-\alpha)} [t^{1-\alpha} - (t-\tau)^{1-\alpha}] \tag{12}$$

Thus

$$\begin{aligned} T_n^{'}(n) &= \frac{1^{1-\alpha} - 0^{1-\alpha}}{\Gamma(2-\alpha)} T^{1-\alpha} \\ T_n^{'}(n-1) &= \frac{2^{1-\alpha} - 1^{1-\alpha}}{\Gamma(2-\alpha)} T^{1-\alpha} \\ &\cdots \\ T_n^{'}(1) &= \frac{n^{1-\alpha} - (n-1)^{1-\alpha}}{\Gamma(2-\alpha)} T^{1-\alpha} \end{aligned} \tag{13}$$

The interpretation of discrete fractional order derivatives is the derivatives of fractional $(1-\alpha)$ order integrals $\int_0^{nT} f(\tau) dg_t^{'}(\tau)$. Namely, it can be understood geometrically as the changing ratio of the “scaled integral area” due to the sampling time scaling property.

Clearly, when the orders are integers, the sampling time scaling effect disappears which means in discrete domain FOC is also a generalization and “interpolation” of the integer order control theory.

B. Truncated discretization

Based on (8) and (13), it is easy to give the discrete equivalent of fractional α order integral or derivative controllers as follows:

$$Z\{D^\alpha[x(t)]\} \approx \frac{1}{T^\alpha} \sum_{j=0}^{\infty} c_j z^{-j} \tag{14}$$

For integral controllers ($\alpha < 0$), coefficients c_j are

$$\begin{aligned} c_0 &= \frac{1}{2\Gamma(1+|\alpha|)} \\ c_j &= \frac{(j+1)^{|\alpha|} - (j-1)^{|\alpha|}}{2\Gamma(1+|\alpha|)}, \ j \geq 1 \end{aligned} \tag{15}$$

and the coefficients of derivative controllers ($\alpha > 0$) are

$$\begin{aligned} c_0 &= \frac{1}{2\Gamma(2-\alpha)} \\ c_1 &= \frac{2^{1-\alpha} - 1}{2\Gamma(2-\alpha)} \\ c_j &= \frac{1}{2\Gamma(2-\alpha)} \left[(j+1)^{1-\alpha} - j^{1-\alpha} \right. \\ &- \left. (j-1)^{1-\alpha} - (j-2)^{1-\alpha} \right], \ j \geq 2 \end{aligned} \tag{16}$$

The semi-log chart of Fig. 13 shows the scaling sampling time versus the step under execution in (8). The observation of the chart gives that the scaled sampling time near ”staring point” t_0 is small enough to be neglected or ”forgotten” for large t. Therefore, the m-term truncated form of (14) can be used as a direct discretization method for realizing fractional order controllers:

$$Z(D^\alpha[x(t)]) \approx \frac{1}{T^\alpha} \sum_{j=0}^{m} c_j z^{-j} \tag{17}$$

Obviously, in order to have a better approximation, longer memory length m is preferable.

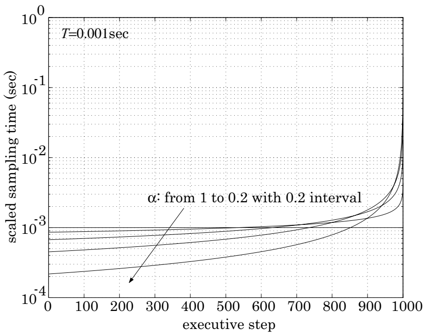

Fig. 13. Scaled sampling time vs. executive step

V. EXPERIMENTAL RESULTS

The experimental torsional system is controlled by a PC with 1.6GHz Pentium IV CPU and 512M memory. The control programs are written in RTLinux C threads which can be executed with strict timing requirement of control sampling time. Experiments on PID^k speed control are carried out with sampling time T=0.001sec. Two encoders (8000$pulse/rev$) are used as rotation speed sensors with coarse quantization $\pm 0.785 rad/sec$.

Firstly, integer order PID speed control experiment is carried out. As depicted in Fig. 14, the PID control system can achieve satisfactory response when the backlash angle is adjusted to zero degree ($\delta = 0$), while severe vibration occurs due to the existence of backlash non-linearity (see $\delta = 0.5$ case). This experimental result is consistent with the analysis in section III-A.

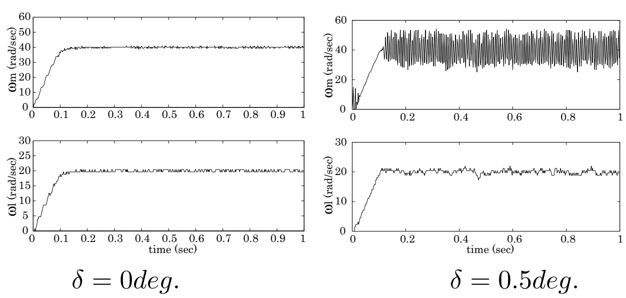

Fig. 14. Time responses of the integer order PID control

Experiments of torsional system's PID^k speed control are carried out with different D controller's order k and memory length m. Figures 15 and 16 depict the experimental results of fractional order PID^k control with 0.2, 0.4, 0.6, 0.8 order D^k controllers and memory length of $m = 5$ and $m = 100$. The control system's stability and robustness against backlash non-linearity are obviously improved and the severe backlash vibration in integer order PID control case is suppressed. It can be seen in Fig. 15 and Fig. 16 that better approximation and performances can be achieved with longer memory length; while even taking short memory length such as $m = 5$ can also give satisfactory performances. The intermittent tiny vibrations in lower order 0.6, 0.4 and 0.2 cases are due to their relative high gains near gear backlash vibration mode in open-loop frequency responses.

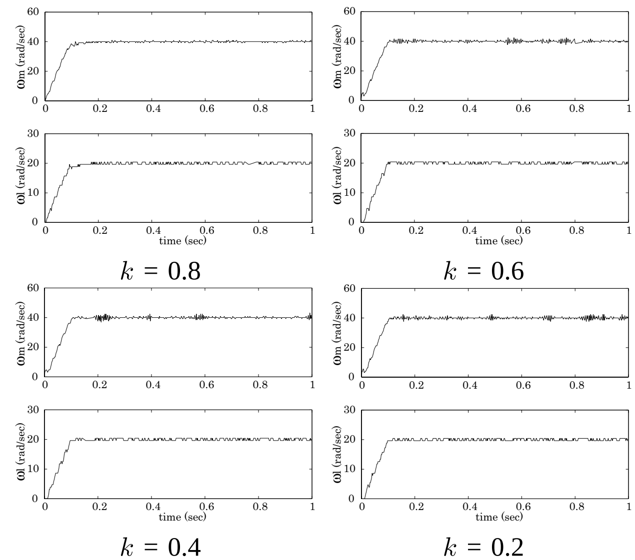

Fig. 15. Time responses of PID^k control ($m = 5$)

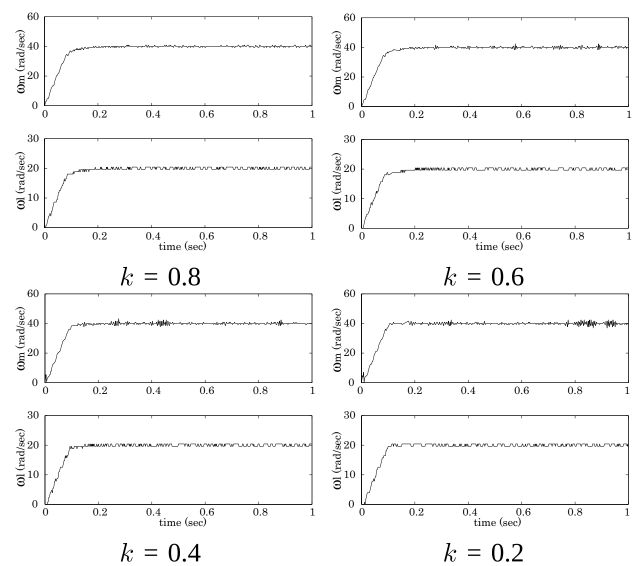

Fig. 16. Time responses of PID^k control ($m = 100$)

It is interesting to find the time responses of the fractional order PID^k control systems also show the "interpolation" characteristic between their integer order counterparts. As depicted in Fig. 17, the time responses of $PID^{0.99}$ and $PID^{0.01}$ closely resemble PID^1 and PID^0's time responses, while this experimental result is natural since the orders are nearly same. The "interpolation" characteristic is one of main points to understand the superiority of FOC as providing more flexibility for designing robust control systems. At the same time, the experimental consistency with the logicality also verifies the good approximation of the realization method based on the sampling time scaling property.

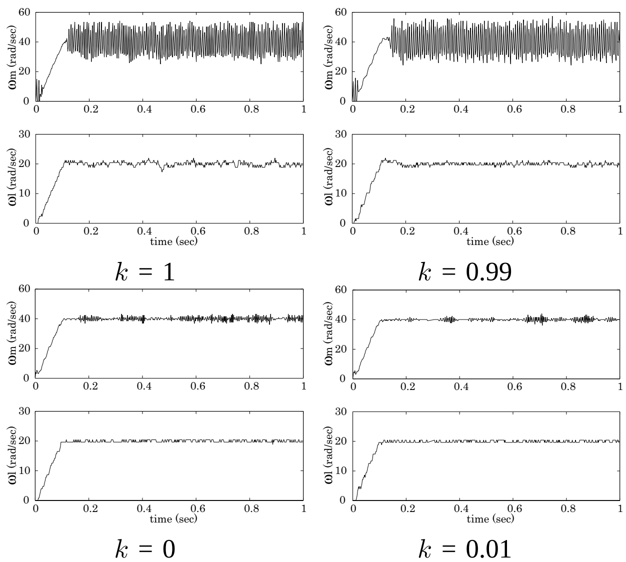

Fig. 17. Continuity of PID^k control's time responses (the fractional order controllers are realized with $m = 100$)

VI. CONCLUSIONS

In this paper, a fractional order PID^k controller is applied to torsional system's backlash vibration suppression control. The designed fractional D^k controller is realized by digital computer based on the sampling time scaling property. Experimental results show PID^k control system's improved robustness against backlash non-linearity and good approximation of the realization method. Based on classical integer order PID controller, by letting D controller's order be fractional order k, the control system's frequency response can be adjusted directly. This flexibility leads to more straightforward design and less tuning effort in real industrial applications. Even having a little higher hardware demand, cheaper design cost and superior robustness of FOC demonstrated in this paper still highlight its promising aspects. Rapid development of computational power also makes fractional order controller's implementation not really problematic. On the other hand, applying FOC concept to motion control is still in a primitive stage. Future researches on FOC theory and its applications to more complex control problems are needed.

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