TRADE-OFF ADJUSTMENT OF FRACTIONAL ORDER LOW-PASS FILTER FOR VIBRATION SUPPRESSION CONTROL OF TORSIONAL SYSTEM

Chengbin Ma * **Yoichi Hori** **

* *Department of Electrical Engineering, School of Engineering, the University of Tokyo, 4-6-1 Komaba, Meguro-ku, Tokyo, Japan 153-8505*
** *Electrical Control System Engineering, Information and System Division, Institute of Industrial Science, the University of Tokyo, 4-6-1 Komaba, Meguro-ku, Tokyo, Japan 153-8505*

Abstract: This paper proposes a fractional order low-pass filter $\frac{1}{(Ts+1)^\alpha}$ for adjusting the trade-off between stability margin loss and the strength of vibration suppression, in which order α can not only be integer but also be any real number. The necessity of this trade-off adjustment is common and natural in oscillatory system's control. For such kind of systems, classical PI control with fractional order low-pass filter $\frac{1}{(Ts+1)^\alpha}$ could be a general solution. As a novel approach, by letting the order α of low-pass filter $\frac{1}{(Ts+1)^\alpha}$ be fractional, control system's frequency response can be adjusted easily. This superiority of Fractional Order Control (FOC) leads to a clear-cut design that is desired in engineering applications. The trade-off in oscillatory system control can be adjusted directly through FOC approach. In this paper, torsional system's speed control is used as a case study for an experimental verification of FOC's theoretical superiority. For implementation of fractional order low-pass filter, broken-line approximation method is applied. Design process and experimental results demonstrate that a "simple & clear-cut design" can be achieved by introducing FOC concept.

Keywords: Oscillatory system, Trade-off, Adjustment, Fractional order low-pass filter

1. INTRODUCTION

The concept of Fractional Order Control (FOC) means controlled systems and/or controllers are described by fractional order differential equations. Expanding derivatives and integrals to fractional orders has a firm and long standing theoretical foundation. Leibniz mentioned this concept in a letter to L'Hospital over three hundred years ago in 1695 and the earliest more or less systematic studies have been made in the beginning and middle of the 19th century by Liouville, Holmgren and Riemann (Oldham and Spanier, 1974), (Podlubny, 1999a). As one of its applications in control engineering, FOC was introduced by Tustin for the position control of massive objects half a century ago, where actuator saturation requires sufficient phase margin around and below the critical point (Tustin, *et al.*, 1956). Some pioneering works were also done in 60's

(Manabe, 1960). However the FOC concept was not widely incorporated into control engineering mainly due to the conceptually difficult idea of taking fractional order, the existence of so few physical applications and the limited computational power available at that time (Axtell and Bise, 1990).

In the last few decades, researchers pointed out that fractional order differential equations could model various materials more adequately than integer order ones and provide an excellent tool for describing dynamic processes (Vinagre and Feliú, 1998). The fractional order models need fractional order controllers for more effective control of the dynamic systems (Podlubny, 1999b). At the same time, letting control order be fractional can give a straightforward way to adjust control system's frequency response. This great flexibility makes it possible to design more robust control system with less control parameters. The superiorities of FOC in modeling and control design motivated renewed interest in various applications of FOC (Machado, 1997), (Oustaloup and Moreau, 1998), (Petras and Vinagre, 2001). With the rapid development of computer performances, modeling and realization of the FOC systems also became possible and much easier than before.

Despite FOC's promising aspects in control modeling and design, FOC research is still at its primary stage. Parallel to the development of FOC theories, applying FOC to various control problems is also crucially important and should be one of top priority issues. The authors believe that designing FOC systems should be clear-cut and there is no reason that we don't make good use of extremely well developed classical Integer Order Control (IOC) theories.

Based on these basic considerations, in this paper, the authors introduce a fractional order version of low-pass filter $\frac{1}{(Ts+1)^\alpha}$ to achieve a clear-cut adjustment of the trade-off between stability margin loss and the strength of vibration suppression in speed control of torsional system. The necessity of this trade-off adjustment is common and natural in oscillatory system's control (Chen, *et al.*, 2003). For such kind of systems, classical PI control with fractional order low-pass filter $\frac{1}{(Ts+1)^\alpha}$ can be a general solution. This paper contributes to the verification of the above hypothesis on an experimental basis.

2. THE TESTING BENCH

The testing bench of torsional system is depicted in Fig. 1, which is a typical oscillatory system. Two flywheels are connected with a long torsional shaft; while driving force is transmitted from driving servomotor to the shaft by gears with gear ratio 1:2. Some system parameters are adjustable, such as gear inertia, load inertia, shaft elastic coefficient and gear backlash angle. The encoders and tacho-generators are used as position and rotation speed sensors.

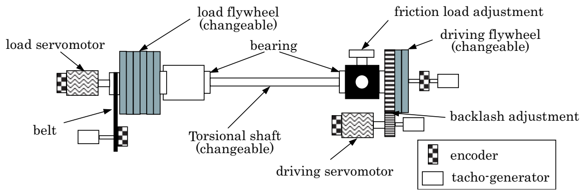


Fig. 1. Experimental setup of the torsional system

The simplest model of the testing bench with gear backlash is three-inertia model, as depicted in Fig. 2 and Fig. 3, where J_m, J_g and J_l are driving motor, gear (driving flywheels) and load inertias, K_s shaft elastic coefficient, ω_m and ω_l motor and load rotation speeds, T_m input torque and T_l disturbance torque. The gear backlash non-linearity is described by the classical dead zone models in which the shaft is modeled as a pure spring with zero damping (Nordin and Gutman, 2002). Frictions between components are neglected due to their small values.

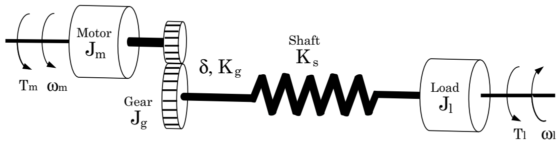


Fig. 2. Torsional system's three-inertia model

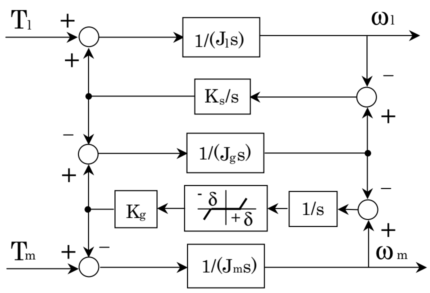


Fig. 3. Block diagram of the three-inertia model

Parameters of the experimental torsional system are shown in Table. 1 with a backlash angle δ of $\pm 0.6deg$. Open-loop transfer function from T_m to ω_m is in the following form:

$$P_{3m}(s) = \frac{(s^2 + \omega_{h1}^2)(s^2 + \omega_{h2}^2)}{J_m s(s^2 + \omega_{o1}^2)(s^2 + \omega_{o2}^2)} \quad (1)$$

where ω_{o1} and ω_{o2} are resonance frequencies while ω_{h1} and ω_{h2} are anti-resonance frequencies. ω_{o1} and ω_{h1} correspond to torsion vibration mode; while ω_{o2} and ω_{h2} correspond to gear backlash vibration mode (see Fig. 4).

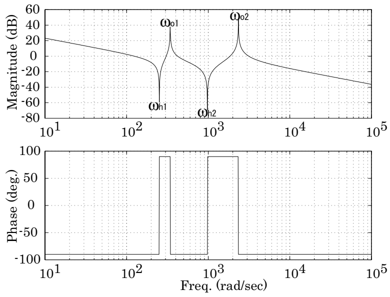


Fig. 4. Bode plots of the three-inertia model

Table 1. Parameters of the three-inertia system

J_m (Kgm^2)	J_g (Kgm^2)	J_l (Kgm^2)	K_g (Nm/rad)	K_s (Nm/rad)
0.0007	0.0034	0.0029	3000	198.49

3. NECESSITY OF TRADE-OFF ADJUSTMENT

As mentioned by Ma and Hori (2004), a well designed set-point-I PI controller can give a satisfactory performance for speed control in nominal case (see Fig. 5 and Fig. 6). The PI controller is designed by Coefficient Diagram Method (CDM) with $K_i = 119.78$ and $K_p = 1.6187$ (Hori, 1995) (Manabe, 1998).

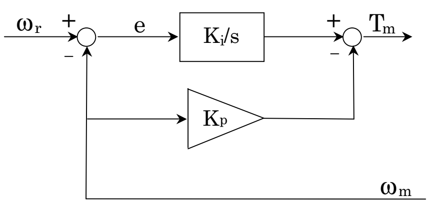


Fig. 5. Set-point-I PI controller

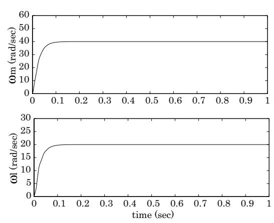


Fig. 6. Simulation results with nominal three-inertia model

For nominal three-inertia model $P_{3m}(s)$, the close-loop transfer function of integer order PI control system from ω_r to ω_m is

$$G_{close}(s) = \frac{C_I(s)P_{3m}(s)}{1 + C_I(s)P_{3m}(s) + C_P(s)P_{3m}(s)} \quad (2)$$

where $C_I(s)$ is I controller and $C_P(s)$ is P controller in minor loop; therefore $G_{close}(s)$ is stable if and only if $G_l = C_I(s)P_{3m}(s) + C_P(s)P_{3m}(s)$ has positive gain margin and phase margin. At the same time, for torsional system's speed control, suppressing vibration caused by the gear backlash must be concerned.

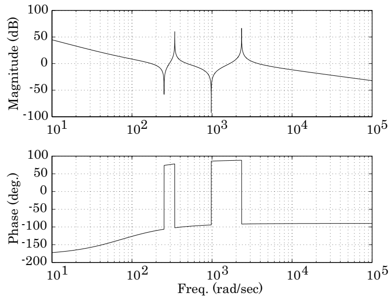


Fig. 7. Bode plots of $G_l(j\omega)$ with PI controller

As depicted in Fig. 7, the PI speed control system has enough stability margin; while in order to recover some vibration performance, additional factors with negative slope and phase-lag are needed. However introducing these factors will simultaneously lead to phase margin loss. Namely, there exists a trade-off between stability margin loss and the strength of vibration suppression in the testing torsional system's speed control.

4. FRACTIONAL ORDER FILTER

In order to achieve a proper controller, which is neither conservative nor aggressive, redesigning the PI controller or applying other control methods can be options; while in this paper, a fractional order low-pass filter $\frac{1}{(Ts+1)^\alpha}$ is introduced (see Fig. 8). The trade-off between stability margin loss and the strength of vibration suppression can be adjusted easily by choosing proper fractional order α only, as depicted in Fig. 9. T will give control system enough large band width for a fast time response. Here considering the frequency range of torsion vibration mode, T is taken as 0.005(=1/200).

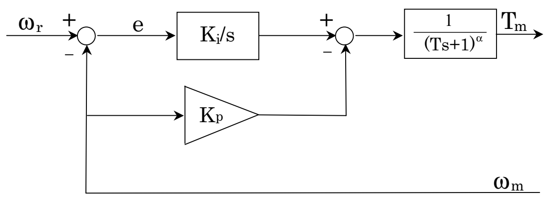


Fig. 8. PI controller with fractional order low-pass filter

5. REALIZATION METHOD

Design control system by FOC approach is clear-cut. However, for realizing designed fractional order controller, it is not so. Due to fractional order systems' infinite dimension, proper approximation by finite difference equation is needed. Since FOC system's frequency response is actually exactly

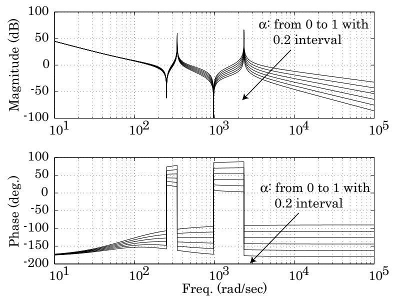


Fig. 9. Bode plots of $G_l(j\omega)$ with fractional order low-pass filters

known. It is natural to approximate fractional order controllers by frequency domain approaches.

In this paper, a broken-line approximation method is introduced to approximate $\frac{1}{(Ts+1)^{\alpha}}$ in frequency range $[\omega_b,\ \omega_h]$, where $T = \frac{1}{\omega_b}$. ω_h is taken as 10^4 to give an enough frequency range for a good approximation. Let

$$\left(\frac{\frac{s}{\omega_h}+1}{\frac{s}{\omega_b}+1}\right)^{\alpha} \approx \prod_{i=0}^{N-1} \frac{\frac{s}{\omega_i^{'}}+1}{\frac{s}{\omega_i}+1} \tag{3}$$

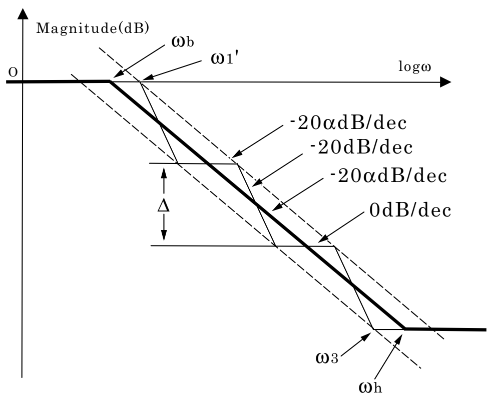


Fig. 10. An example of broken-line approximation $(N = 3)$

Based on Fig. 10, two recursive factors ζ and η are introduced to calculate ω_i and $\omega_i^{'}$:

$$\zeta = \frac{\omega_i^{'}}{\omega_i},\ \eta = \frac{\omega_{i+1}}{\omega_i^{'}} \tag{4}$$

Since

$$\omega_0 = \eta^{\frac{1}{2}}\omega_b,\ \omega_{N-1}^{'} = \eta^{-\frac{1}{2}}\omega_h \tag{5}$$

Therefore

$$\zeta\eta = \left(\frac{\omega_h}{\omega_b}\right)^{\frac{1}{N}} \tag{6}$$

with

$$\omega_i = (\zeta\eta)^i\omega_0,\ \omega_i^{'} = \zeta(\zeta\eta)^i\omega_0 \tag{7}$$

The frequency-band fractional order controller has $-20\alpha dB/dec$ gain slope, while the integer order factors $1\Big/(\frac{s}{\omega_i^{'}}+1)$ have $-20dB/dec$ slope. For the same magnitude change Δ:

$$-20\alpha = \frac{\Delta}{log\zeta + log\eta},\ -20 = \frac{\Delta}{log\zeta} \tag{8}$$

Thus

$$(\zeta\eta)^{\alpha} = \zeta \tag{9}$$

Therefore ζ and η can be expressed respectively by

$$\zeta = \left(\frac{\omega_h}{\omega_b}\right)^{\frac{\alpha}{N}},\ \eta = \left(\frac{\omega_h}{\omega_b}\right)^{\frac{1-\alpha}{N}} \tag{10}$$

Finally

$$\omega_i = \left(\frac{\omega_h}{\omega_b}\right)^{\frac{i+\frac{1}{2}-\frac{\alpha}{2}}{N}}\omega_b,\ \omega_i^{'} = \left(\frac{\omega_h}{\omega_b}\right)^{\frac{i+\frac{1}{2}+\frac{\alpha}{2}}{N}}\omega_b \tag{11}$$

Figure. 11 shows the Bode plots of ideal frequency-band case ($\alpha = 0.4,\ \omega_b = 200Hz,\ \omega_h = 1000Hz$) and its 1st-order, 2nd-order and 3rd-order approximations by broken-line approximation method. Even taking $N = 2$ can give a satisfactory accuracy in frequency domain. For digital implementation, the bilinear transformation method is used in this paper.

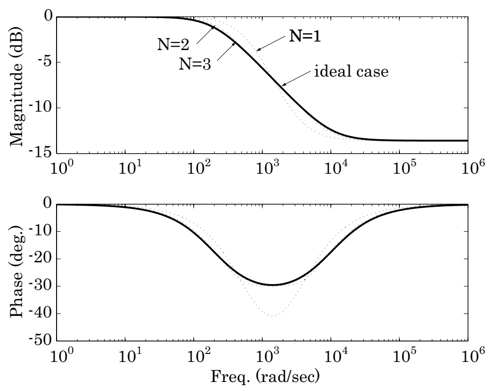


Fig. 11. Bode plots of ideal case, 1st, 2nd and 3rd-order approximations

6. EXPERIMENTAL RESULTS

As depicted in Fig. 12, the experimental torsional system is controlled by a PC with 1.6GHz Pentium IV CPU and 512M memory. Control programs are written in RTLinux C threads which can be executed with strict timing requirement of control sampling time. A 12-bit AD/DA multi-functional board is used whose conversion time per channel is $10\mu sec$.

Experiments are carried out with sampling time T=0.001sec and 2nd-order broken-line approximation ($N = 2$). Two encoders ($8000pulse/rev$)

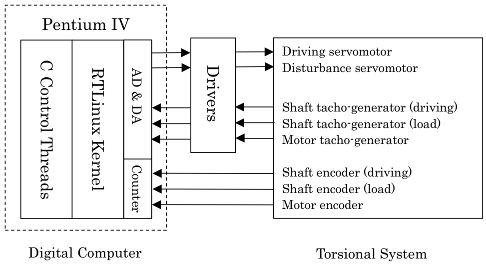
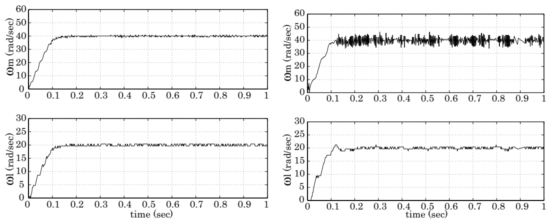


Fig. 12. Digital control system of the experimental setup

are used as rotation speed sensors with coarse quantization $\pm 0.785 rad/sec$.

Since the driving servomotor's input torque command T_m has a limitation of maximum ± 3.84 Nm, K_i is reduced to 18.032 by trial-and-error to avoid large over-shoot caused by the saturation. Firstly, integer order PI speed control experiment is carried out. As depicted in Fig. 13 the PI control system can achieve satisfactory response when the backlash angle is adjusted to zero degree ($\delta = 0$); while persistent vibration occurs when gear backlash non-linearity exists (see $\delta = 0.6$ case).

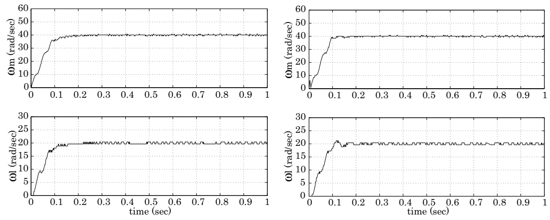


(a) $\delta = 0deg.$ (b) $\delta = 0.6deg.$

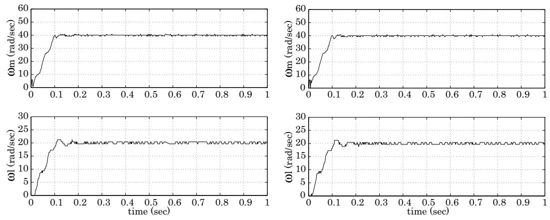
Fig. 13. Experimental results of the integer order PI control

Figure 14 depicts the experimental results with different α order filters. Vibration occurred in PI-only control is effectively suppressed by introducing fractional order low-pass filter $\frac{1}{(Ts+1)^\alpha}$. In those results, taking α as 0.2 gives best time response with improved vibration suppression performance. For other higher α order cases, even the vibration is suppressed, their time responses are not such satisfied due to more phase margin loss. This observation gives that, by FOC approach, it is more clear-cut to adjust the trade-off between stability margin loss and strength of vibration suppression.

In order to verify whether the fractional order filter can give a continuous tuning of the trade-off, the time responses of $\alpha = 0.01$ and $\alpha = 0.99$ cases are also experimented. As depicted in Fig. 15, the results show a good continuity. Attention should be paid toward the reasons for vibrations in two cases. Poor vibration suppression performance causes vibration in $\alpha = 0.01$ case; while nearly zero phase margin in $\alpha = 0.99$ case leads to the severe vibration with lower frequency



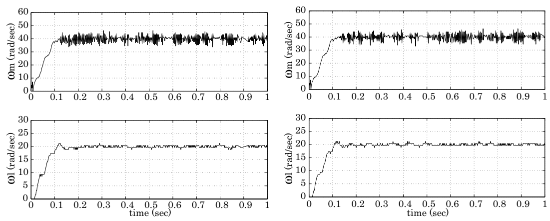
(a) $\alpha = 0.2$ (b) $\alpha = 0.4$



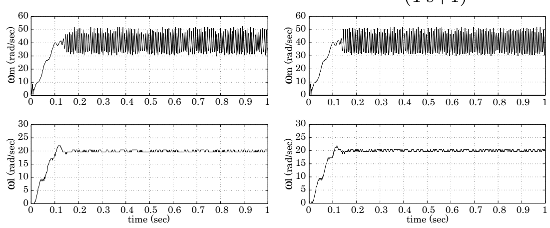
(c) $\alpha = 0.6$ (d) $\alpha = 0.8$

Fig. 14. Experimental results with fractional order $\frac{1}{(Ts+1)^\alpha}$ filter

and larger amplitude. Namely, the reason for the second case is due to its poor relative stability. A proper fractional order α can give a better trade-off between these two extreme cases.



(a) PI-only (b) PI+$\frac{1}{(Ts+1)^{0.01}}$



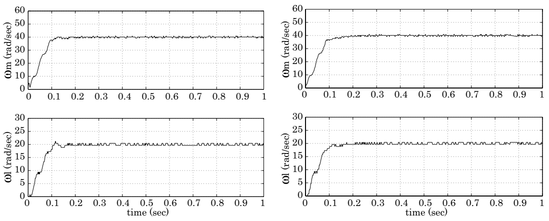
(c) PI+$\frac{1}{(Ts+1)^{0.99}}$ (d) PI+$\frac{1}{Ts+1}$

Fig. 15. Continuity of Experimental results with different fractional order α

Figure. 16 depicts experimental results with the 1st-order and 3rd-order approximation of broken-line method ($\alpha = 0.2$). Even taking 1st-order approximation can give a relatively good performance.

7. CONCLUSIONS

In this paper, a classical PI controller with fractional order low-pass filter $\frac{1}{(Ts+1)^\alpha}$ is proposed to give a straightforward trade-off adjustment between the control system's stability margin loss and the strength of vibration suppression. In oscillatory system control, this kind of trade-off is a common problem. As shown in the above theoretical analysis and experimental results, by introducing FOC concept, we can design control system in



(a) 1st-order (b) 3rd-order

Fig. 16. Experimental results with different approximation orders

a clear-cut way since control system's frequency response can be adjusted easily to desired shape with few control parameters. Namely, the tuning knob can be reduced significantly compared to high-order transfer functions obtained by classical IOC approaches.

At the same time, it can be seen using fractional order controller is a general method to trade off inconsistent control demands, which is not limited to the specific problem. "Simple & clear-cut design" can be achieved by expending controller's order to fractional.

On the contrary to FOC control design, the implementation of fractional order controllers is not such direct. Some proper approximations are needed. However, as verified in experimental results, the implementation issue actually should not be problematic.

FOC is not an abstract concept, but a natural and powerful expansion of the well-developed IOC. Knowledge and design tools developed in IOC can still be made good use of in FOC research, as demonstrated in this paper. For example, upgrading traditional PID controller by introducing fractional order factors, such as fractional order I^α, D^β controllers or fractional order filters, could give a more effective control of complex dynamics. It is interesting to find that in the experiments the 1st-order approximation can also have a relative good performance (see Fig. 16). This filter is actually a simple one order controller:

$$0.45731\frac{(s+2091)}{(s+956.4)} \quad (12)$$

The authors do believe some well-designed IOC system might in fact be a unconscious approximation of FOC system. If this hypothesis can be established, FOC's superiorities in control field would be further verified.

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