Paper

Time-Domain Evaluation of Fractional Order Controllers' Direct Discretization Methods

Chengbin Ma* Student Member
Yoichi Hori* Member

Fractional Order Control (FOC), in which the controlled systems and/or controllers are described by fractional order differential equations, has been applied to various control problems. Though it is not difficult to understand FOC's theoretical superiority, realization issue keeps being somewhat problematic. Since the fractional order systems have an infinite dimension, proper approximation by finite difference equation is needed to realize the designed fractional order controllers. In this paper, the existing direct discretization methods are evaluated by their convergences and time-domain comparison with the baseline case. Proposed sampling time scaling property is used to calculate the baseline case with full memory length. This novel discretization method is based on the classical trapezoidal rule but with scaled sampling time. Comparative studies show good performance and simple algorithm make the Short Memory Principle method most practically superior. The FOC research is still at its primary stage. But its applications in modeling and robustness against non-linearities reveal the promising aspects. Parallel to the development of FOC theories, applying FOC to various control problems is also crucially important and one of top priority issues.

Keywords: Fractional Order Controller, Direct Discretization methods, Time-Domain Evaluation

1. Introduction

The concept of Fractional Order Control (FOC), in which the controlled systems and/or controllers are described by fractional order differential equations, is by no means new. In fact, it has a long history. The concept was firstly introduced by *Tustin* for the position control of massive objects half a century ago, where the actuator saturation requires sufficient phase margin around and below the crossover frequency [(1)].

However, FOC was not widely incorporated into control engineering mainly due to the conceptually difficult idea of taking fractional order, the existence of so few physical applications and the limited computational power available at that time [(2)]. In last few decades, researchers pointed out that fractional order differential equations could model various materials more adequately than integer order ones and provide an excellent tool for describing dynamic processes [(3) (4) (5)]. The fractional order models need fractional order controllers for more effective control of the dynamic systems [(6)]. This necessity motivated the renewed interest in various applications of FOC [(7) (8) (9) (10)]. Thanks to the rapid development of computational power, modeling and realizing FOC systems also became much easier than before.

By changing FOC controller's fractional order, control system's frequency response can be adjusted directly and continuously. This advantage can lead to more straightforward design of robust control systems against uncertainties. Though it is not difficult to understand FOC's theoretical superiority, realization issue keeps being somewhat problematic and perhaps is one of the most doubtful points for applying FOC. Since the fractional order systems have an infinite dimension, proper approximation by finite difference equation is needed to realize the designed fractional order controllers.

Frequency-band fractional order controller can be realized by broken-line approximation in frequency-domain, but further discretization is required for this method [(11)]. As to direct discretization, several methods have been proposed such as Short Memory Principle [(4)], Tustin Taylor Expansion [(12)] and Lagrange Function Interpolation method [(8)], while all the approximation methods need truncation of the expansion series. How to determine the baseline case, which is reliable and easy to understand, is essentially important for the evaluation of the proposed methods in time-domain, especially from the viewpoint of engineering.

At the same time, it is well known that the discrete integer order controllers have clear time-domain interpretation as changing ratio or the area of sampled input to time, which significantly simplify their use in various applications. However all the above direct discretization methods for fractional order controllers have a common weak point of lacking clear time-domain interpretation. A clear time-domain interpretation is crucial for the applications of FOC.

The authors proposed a novel and clear time-domain interpretation of discrete fractional order controllers as having sampling time scaling property [(13)]. In this paper, this interpretation is used to achieve a reliable and easy method for calculating the baseline case for discrete FOC control systems. With the established base-

* Electrical Control System Engineering, Information & System Division, Institute of Industrial Science, the University of Tokyo, 4-6-1 Komaba, Meguro-ku, Tokyo 153-8505

line, the discretization methods are evaluated in time-domain. The article is organized as follows: in section 2, the mathematical definitions of fractional order calculus are introduced; in section 3, some typical existing direct discretization methods are reviewed; in section 4, a novel and reliable discretization method is proposed based on the discrete fractional order controllers' sampling scaling property; in section 5, comparative studies are carried out with the discretization methods and the baseline case calculated by the proposed novel method; finally, in section 6, conclusions are drawn.

2. Mathematical Definitions

The mathematical definition of fractional derivatives and integrals has been a subject of several different approaches [3] [4]. The most frequently encountered definition is called Riemann-Liouville definition, in which the fractional α order integrals are defined as

$$ {}_{t_0}I_t^{\alpha} f(t) := {}_{t_0}D_t^{-\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t-\xi)^{\alpha-1} f(\xi) d(\xi) \quad (1) $$

while the definition of fractional order derivatives is

$$ {}_{t_0}D_t^{\alpha} f(t) = \frac{d^{\gamma}}{dt^{\gamma}} \left[{}_{t_0}D_t^{-(\gamma-\alpha)} f(t) \right] \cdots\cdots\cdots\cdots (2) $$

where

$$ \Gamma(x) \equiv \int_0^{\infty} y^{x-1} e^{-y} dy \cdots\cdots\cdots\cdots (3) $$

is the Gamma function, t_0 and t are limits and α ($\alpha > 0$ and $\alpha \in R$) is the order of the operation. γ is an integer that satisfies $\gamma - 1 < \alpha < \gamma$.

The other approach is Grünwald-Letnikov definition:

$$ {}_{t_0}D_t^{\alpha} f(t) = \lim_{\substack{h \to 0 \\ nh = t - t_0}} h^{-\alpha} \sum_{r=0}^{n} (-1)^r \begin{pmatrix} \alpha \\ r \end{pmatrix} f(t - rh) \quad (4) $$

where binomial coefficients are

$$ \begin{pmatrix} \alpha \\ 0 \end{pmatrix} = 1, \begin{pmatrix} \alpha \\ r \end{pmatrix} = \frac{\alpha(\alpha-1)\ldots(\alpha-r+1)}{r!} \quad (5) $$

3. Existing Discretization Methods

3.1 Short memory principle For simplification, the controller is discrete fractional α order derivative ($0 < \alpha < 1$) or integral ($-1 < \alpha < 0$).

The discretization method is based on the observation that for Grünwald-Letnikov definition, the values of the binomial coefficients near the "starting point" $t = 0$ is small enough to be neglected or "forgotten" for large t. Therefore the principle takes into account the behavior of $x(t)$ only in the "recent past", i.e., in the interval $[t - L,\ t]$, where L is the length of "memory":

$$ {}_0D_t^{\alpha} x(t) \approx {}_{t-L}D_t^{\alpha} x(t), (t > L) \cdots\cdots\cdots\cdots (6) $$

Based on approximation of the time increment h through the sampling time T, the discrete equivalent of the fractional order α derivative is given by

$$ Z\{D^{\alpha}[x(t)]\} \approx \left(\frac{1}{T^{\alpha}} \sum_{j=0}^{m} c_j z^{-j} \right) X(z) \cdots\cdots (7) $$

where $m = [L/T]$ and the coefficients c_j are

$$ c_0 = 1, $$

$$ c_j = (-1)^j \begin{pmatrix} \alpha \\ j \end{pmatrix} = \frac{j - \alpha - 1}{j} \cdot c_{j-1}^{\alpha} \cdots\cdots\cdots (8) $$

3.2 Tustin taylor expansion The direct discretization can also be achieved by using the well-known Tustin operator or trapezoidal rule as a generation function as follows:

$$ Z\{D^{\alpha}[x(t)]\} \approx \left(\frac{2}{T} \frac{1 - z^{-1}}{1 + z^{-1}} \right)^{\alpha} X(z) \cdots\cdots\cdots (9) $$

Taylor expansion of the fractional α order Tustin operator gives

$$ \left(\frac{2}{T} \frac{1 - z^{-1}}{1 + z^{-1}} \right)^{\alpha} = \frac{1}{T^{\alpha}} \sum_{j=0}^{\infty} c_j z^{-j} \cdots\cdots\cdots\cdots (10) $$

Here the coefficients c_j are

$$ c_j = \frac{2^{\alpha}}{j!} \left[\left(\frac{1 - x}{1 + x} \right)^{\alpha} \right]^{(j)} \Bigg|_{x=0} \cdots\cdots\cdots\cdots (11) $$

Real implementation of Equ. (9) corresponds to m-term truncated series given by

$$ Z\{D^{\alpha}[x(t)]\} \approx Trunc_m \left[\left(\frac{2}{T} \frac{1 - z^{-1}}{1 + z^{-1}} \right)^{\alpha} \right] X(z) = \left(\frac{1}{T^{\alpha}} \sum_{j=0}^{m} c_j z^{-j} \right) X(z) \cdots\cdots (12) $$

3.3 Lagrange function interpolation For example, quadratic Lagrange interpolation among $x(k-2)$, $x(k-1)$ and $x(k)$ in the interval $0 \le t \le 2T$ results

$$ x(t) = \frac{x(k) - 2x(k-1) + x(k-2)}{2} \left(\frac{t}{T} \right)^2 - \frac{x(k) - 4x(k-1) + 3x(k-2)}{2} \frac{t}{T} + x(k-2) \cdots\cdots\cdots\cdots (13) $$

The α order derivative of t^n is [3]

$$ {}_0D_t^{\alpha}(t^n) = \frac{n! t^{n-\alpha}}{\Gamma(n - \alpha + 1)} \cdots\cdots\cdots\cdots (14) $$

For $t = 2T$, the α order derivative of $x(t)$ is

$$ D^{\alpha} x(t)|_{t=2T} = \frac{1}{T^{\alpha}} \cdot \frac{1}{2^{\alpha}\Gamma(3 - \alpha)} \left[(2 + \alpha) \cdot x(k) - 4\alpha \cdot x(k-1) + \alpha^2 \cdot x(k-2) \right] \quad (15) $$

The z-transformation is

$$ Z\{D^{\alpha} x(t)\} = \frac{1}{T^{\alpha}} \cdot \frac{1}{2^{\alpha}\Gamma(3 - \alpha)} [(2 + \alpha) - 4\alpha z^{-1} + \alpha^2 z^{-2}] X(z) \cdots\cdots\cdots\cdots (16) $$

Therefore, the m-order Lagrange Function Interpolation method can also be re-written in the form:

$$ Z\{D^{\alpha}[x(t)]\} \approx \left(\frac{1}{T^{\alpha}} \sum_{j=0}^{m} c_j z^{-j} \right) X(z) \cdots\cdots (17) $$

4. Proposed Discretization Method

4.1 Sampling time scaling property From the Riemann-Liouville definition, fractional order integral with order between 0 and 1 is

$$ {}_0I_t^\alpha f(t) = \int_0^t f(\tau) dg_t(\tau),\ 0 < \alpha < 1 \cdots\cdots\cdots (18) $$

where

$$ g_t(\tau) = \frac{1}{\Gamma(1+\alpha)}[t^\alpha - (t-\tau)^\alpha] \cdots\cdots\cdots\cdots (19) $$

Let $t := nT$, where T is the sampling time and n is the step currently under execution, then

$$ g_{nT}(kT) = \frac{n^\alpha - (n-k)^\alpha}{\Gamma(1+\alpha)} T^\alpha,\ k = 1, ..., n \cdots (20) $$

Therefore, by sharing the same view of discrete integer order integration rules, the "real" sampling time T of the kth step is

$$ \begin{aligned} T_n(k) &= \Delta g_{nT}(kT) \\ &= g_{nT}(kT) - g_{nT}[(k-1)T] \\ &= \frac{(n-k+1)^\alpha - (n-k)^\alpha}{\Gamma(1+\alpha)} T^\alpha \cdots\cdots\cdots (21) \end{aligned} $$

Thus

$$ \begin{aligned} T_n(n) &= \frac{1^\alpha - 0^\alpha}{\Gamma(1+\alpha)} T^\alpha \\ T_n(n-1) &= \frac{2^\alpha - 1^\alpha}{\Gamma(1+\alpha)} T^\alpha \\ &\cdots \\ T_n(1) &= \frac{n^\alpha - (n-1)^\alpha}{\Gamma(1+\alpha)} T^\alpha \ \cdots\cdots\cdots\cdots (22) \end{aligned} $$

Finally, based on the trapezoidal integration rule

$$ {}_0I_{nT}^\alpha \approx \sum_{k=1}^n \frac{f(kT) + f[(k-1)T]}{2} T_n(k) \ \cdots\cdots (23) $$

and if $T \to 0$, then

$$ {}_0I_{nT}^\alpha = \sum_{k=1}^n \frac{f(kT) + f[(k-1)T]}{2} T_n(k) \ \cdots\cdots (24) $$

From Equ. (22), we can see that the interpretation of discrete fractional order integrals is the "deformation" of their integer order counterparts by internal sampling time scaling (see Fig. 1). By using this interpretation, it becomes transparent to understand that the past values are "forgotten" gradually in discrete fractional order integrals due to their scaled tiny sampling time while in integer order ones all the values are "remembered" with the same weights.

Similarly, discrete fractional order derivatives with order between 0 and 1 is

$$ {}_0D_t^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t \frac{f(\tau)}{(t-\tau)^\alpha} d\tau $$

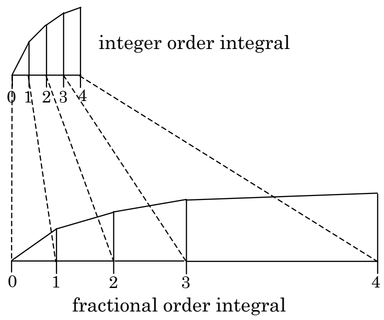


Fig. 1. Fractional order integral's sampling time scaling

$$ = \frac{d[\int_0^t f(\tau) dg_t'(\tau)]}{dt},\ \ 0 < \alpha < 1 \ \cdots\cdots (25) $$

where

$$ g_t'(\tau) = \frac{1}{\Gamma(2-\alpha)}[t^{1-\alpha} - (t-\tau)^{1-\alpha}] \cdots\cdots\cdots (26) $$

Thus

$$ \begin{aligned} T_n'(n) &= \frac{1^{1-\alpha} - 0^{1-\alpha}}{\Gamma(2-\alpha)} T^{1-\alpha} \\ T_n'(n-1) &= \frac{2^{1-\alpha} - 1^{1-\alpha}}{\Gamma(2-\alpha)} T^{1-\alpha} \\ &\cdots \\ T_n'(1) &= \frac{n^{1-\alpha} - (n-1)^{1-\alpha}}{\Gamma(2-\alpha)} T^{1-\alpha} \cdots\cdots (27) \end{aligned} $$

Again based on the trapezoidal integration rule

$$ \int_0^{nT} f(\tau) dg_t'(\tau) \approx \sum_{k=1}^n \frac{f(kT) + f[(k-1)T]}{2} T_n'(k) (28) $$

and if $T \to 0$, then

$$ \int_0^{nT} f(\tau) dg_t'(\tau) = \sum_{k=1}^n \frac{f(kT) + f[(k-1)T]}{2} T_n'(k) (29) $$

The interpretation of discrete fractional order derivatives is the derivatives of fractional $(1-\alpha)$ order integrals $\int_0^{nT} f(\tau) dg_t'(\tau)$. Namely, it can be understood geometrically as the changing ratio of the "scaled integral area" due to the sampling time scaling property, as depicted in Fig. 2.

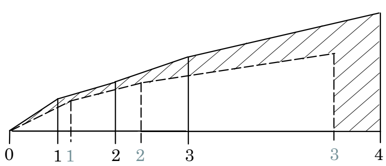


Fig. 2. Changing of the "scaled integral area"

Clearly, when the orders are integers, the sampling time scaling effect disappears which means in discrete domain FOC is also a generalization and "interpolation" of the integer order control theory.

4.2 Full memory length baseline In order to evaluate the discretization methods in time-domain, a reliable baseline case must be calculated in advance. For simulation of FOC systems, using the truncated

Grünwald-Letnikov expansion [6], Mitten-Leffler function [6], Bromwich's integral with a numerical integration and B-spline series expansion [14] can be options. However those methods are either too abstract or too complicated for engineering applications. In this paper, a reliable and easy simulation method is proposed based on the sampling time scaling property, in which the whole past values are memorized. The fractional order controllers are discretized by the classical trapezoidal rule but with scaled sampling time in the method.

Based on Equ. (23) and Equ. (28), it is easy to give the discrete equivalent of the fractional α order integral or derivative controllers as follows:

$$Z\{D^{\alpha}[x(t)]\} \approx \left(\frac{1}{T^{\alpha}}\sum_{j=0}^{\infty} c_j z^{-j}\right) X(z) \cdots\cdots (30)$$

For integral controllers ($\alpha < 0$), coefficients c_j are

$$c_0 = \frac{1}{2\Gamma(1+|\alpha|)}$$
$$c_j = \frac{(j+1)^{|\alpha|} - (j-1)^{|\alpha|}}{2\Gamma(1+|\alpha|)},\ j \geq 1 \cdots\cdots\cdots\cdots (31)$$

And the coefficients of derivative controllers ($\alpha > 0$) are

$$c_0 = \frac{1}{2\Gamma(2-\alpha)}$$
$$c_1 = \frac{2^{1-\alpha} - 1}{2\Gamma(2-\alpha)}$$
$$c_j = \frac{1}{2\Gamma(2-\alpha)}\left[(j+1)^{1-\alpha} - j^{1-\alpha} - (j-1)^{1-\alpha} - (j-2)^{1-\alpha}\right],\ j \geq 2 \cdots\cdots (32)$$

Of course, the m-term truncated form of the proposed simulation method can also be used as a novel direct discretization method for realizing fractional order controllers:

$$Z(D^{\alpha}[x(t)]) \approx \left(\frac{1}{T^{\alpha}}\sum_{j=0}^{m} c_j z^{-j}\right) X(z) \ \cdots\cdots (33)$$

Similarly, the m can be considered to be the approximation's memory length as in the short memory principle method.

5. Comparative Studies

For comparison purposes, one mass position control is introduced as a simple prototype for the case of $J_m = 0.001$ and $K_d = 0.01$ (see Fig. 3). Time responses with fractional order derivative controllers D^{α} are simulated where D^{α} is discretized by using the above direct discretization methods. Sampling time T is taken as $0.001sec$.

Those methods' convergences must be analyzed before applying them to control implementation. The semi-log chart of Fig. 4a shows the amplitude absolute values of the coefficients $|c_j|$ versus term order j when approximating $\alpha = 0.4$ derivative. Short Memory Principle (SMP) and Sampling Time Scaling (STS) methods

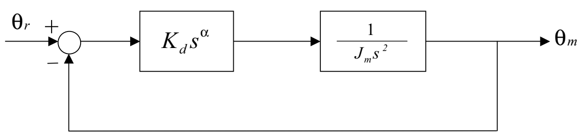
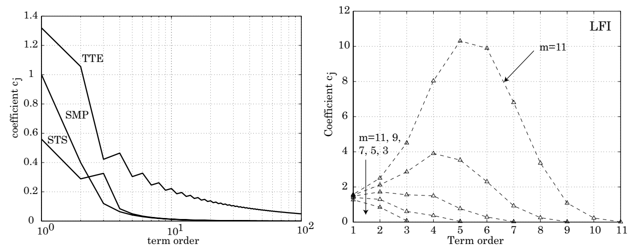


Fig. 3. The position control loop with fractional α order derivative controller

should have similar approximation performances, while the SMP's coefficients converge a little more rapidly than the STS's. The poor convergences of Tustin Taylor Expansion (TTE) and Lagrange Function Interpolation (LFI) methods seem problematic (see Fig. 4a and Fig. 4b).



(a) (b)

Fig. 4. $|c_j|$ versus j when approximating $D^{0.4}$

The baseline time responses with different α order derivative controllers are simulated by the proposed simulation method using the sampling time scaling property. As depicted in Fig. 5, it can be seen clearly that the FOC systems' time responses are an interpolation of the classical integer order ones and can be adjusted continuously by changing order α.

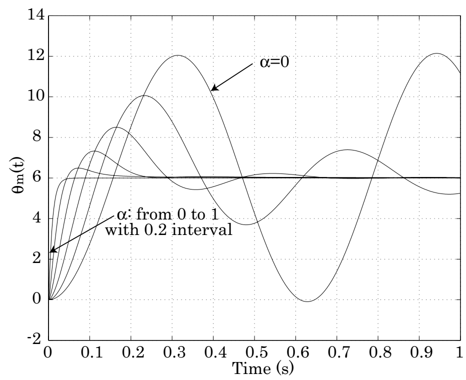
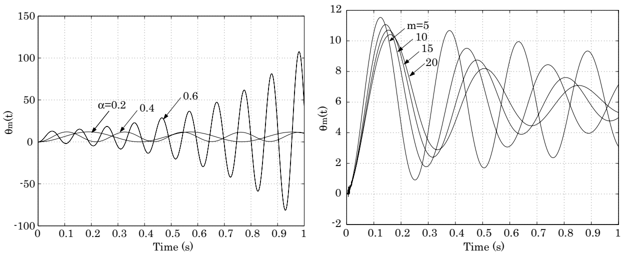


Fig. 5. Time responses with fractional order D^{α} controller

5.1 TTE and LFI methods The simulations of TTE and LFI methods verify the convergence analysis. As depicted in Fig. 6a with approximation order $m = 5$, the TTE method results poor performances. Actually the fractional order controllers realized by high order TTE methods can make control systems unstable, while higher the order better the approximation should be achieved. The time responses of LFI method for $D^{0.4}$ controller are also unsatisfied (see Fig. 6b). In addition the programming complexity of calculating high order Lagrange interpolation and Tustin operator's high order derivative makes the two methods inferior to control applications.



(a) (b)

Fig. 6. Time responses of TTE (a) and LFI (b) methods

5.2 SMP and STS methods In order to investigate the affection of the memory length in SMP and TST methods, a quadratic performance index J is defined in an error function form:

$$J = \int_0^t [f_a(\tau) - f_b(\tau)]^2 d\tau \quad \cdots\cdots\cdots\cdots\cdots\cdots (34)$$

with $t(= 1sec)$ simulation time, $f_a(t)$ time responses of the two approximation cases, $f_b(t)$ the baseline time response. The baseline case is calculated by full memory length STS method. Fig. 7 shows performance index J versus memory length $n(\geq 5)$, in which the fractional order α is from 0.8 to 0.2 with 0.2 interval.

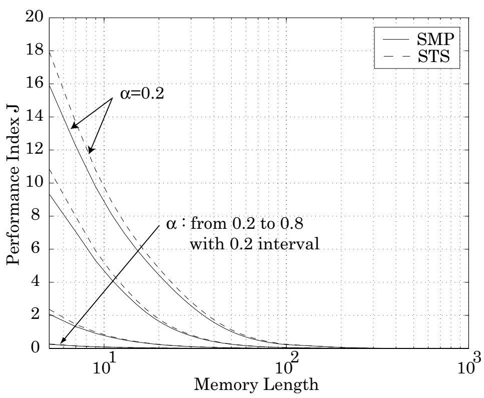


Fig. 7. Performance index versus memory length

The four quantities of the step responses, maximum overshoot, delay time, rise time and settling time, are calculated for both methods. For clearness, only $\alpha = 0.4$ case is plotted in Fig. 8.

As depicted in Fig. 7, clearly the approximation performance is remarkably improved when increasing the memory length from 10 to 100. Between 100 and 1000 memory length, the performance improvement is just slight; while hardware burden increases due to the necessity of storing and processing more data in short time. The step response's quantities plotted in Fig. 8 also show the same observation result. The SMP method has a slightly better approximation than the STS method. The programming of SMP method is also much easier in which c_j can be calculated by simply multiplying c_{j-1} and $(j - \alpha - 1)/j$ together, as shown in Equ. (8). The SMP method is practically superior. When sampling time T is $0.001sec$, taking 100 memory length can have a good approximation (see Fig. 9). With highly-developed computational power, processing 100 sampling data with simple SMP algorithm should not be

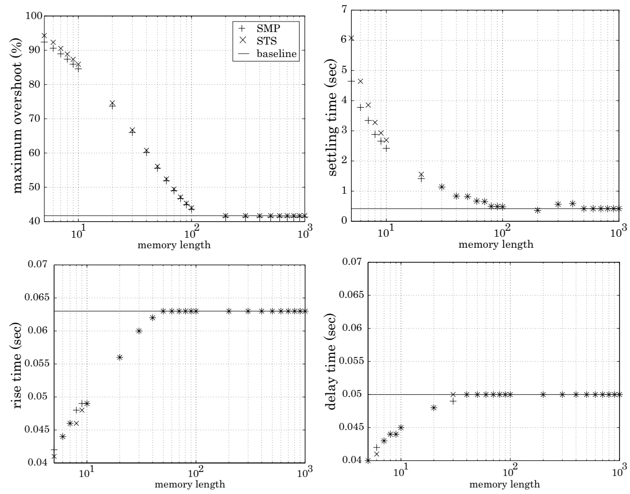


Fig. 8. Time responses' four quantities

problematic in mili-second level for modern digital control systems. In real application, even memorizing 10 past values can also give a good control performance [15]. The necessary memory length, namely how good the approximation is needed, should be decided by the demand of specific control problem.

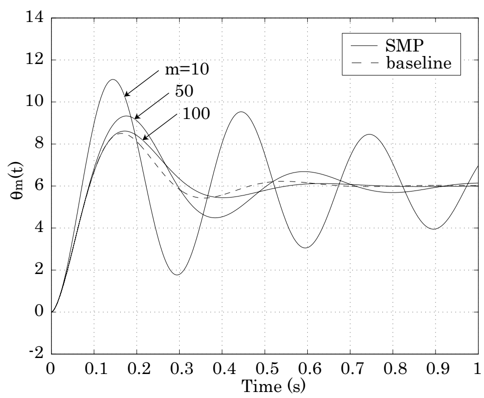


Fig. 9. Time responses with different memory length m $(\alpha = 0.4)$

For the one mass position control with D^α controllers, the open-loop is $\frac{1}{s^{2-\alpha}}$ and its phase margin is $(2-\alpha) \times 90$ degree. A proper phase margin can be easily achieved by choosing fractional order α [1]. Letting α be 0.4 gives the control system a good robustness against saturation non-linearity [16], which is one of the most ordinary non-linear phenomena in control systems. As depicted in Fig. 10, a maximum torque limitation of $\pm 2Nm$ is introduced in the unity feedback control system. Comparison of Fig. 5 and Fig. 11 verifies that the well-approximated fractional order $D^{0.4}$ controllers are remarkably robust against saturation non-linearity. It was found that the fractional order controllers, like PID^α controller, are robust against other non-linearities such as gear backlash [15].

6. Conclusions

In this paper, the sampling time scaling property is used as a reliable and easy method to calculate the baseline case with full memory length. This simulation method is based on the classical trapezoidal rule

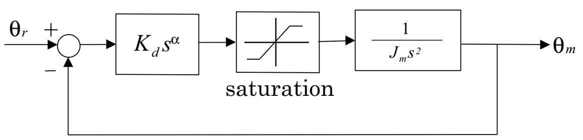


Fig. 10. The position control loop with torque saturation

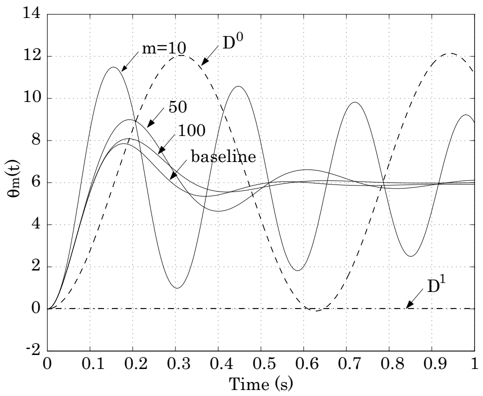


Fig. 11. Robustness of approximated $D^{0.4}$ controller against saturation non-linearity (dash lines are the time responses with integer 1 and 0 order D^α controllers)

but with scaled sampling time. It's truncated form is also proposed as a novel direct discretization method. The existing direct discretization methods are evaluated by their convergences and time-domain comparison with the established baseline case. Comparative studies show the poor performance of TTE and LFI methods. SMP and STS methods have better and similar approximation; while the simple algorithm makes SMP method practically superior. With the baseline case calculated by the proposed simulation method, the original plots of quadratic performance index and the other four quantities give a clear way to evaluate the effect of memory length. The simulation results show remembering 100 past value can achieve a good approximation.

The FOC research is still at its primary stage. But its applications in modeling and robustness against non-linearities reveal the promising aspects. Parallel to the development of FOC theories, applying FOC to various control problems is also crucially important and should be one of top priority issues.

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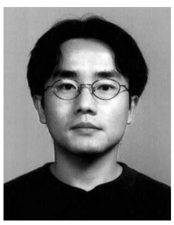
Chengbin Ma (Student Member) was born in Chengcheng county, Shanxi province, China on 21st November 1975. He received the master degree of engineering from Department of Electrical Engineering, the University of Tokyo in September 2001 and is presently a Ph.D. student in the same department.



His doctoral research fields are fractional order control theory and applications to motion control.

Yoichi Hori (Member) received the B.S., M.S. and Ph.D degrees in Electrical Engineering from the University of Tokyo in 1978, 1980 and 1983, respectively. In 1983, he joined the University of Tokyo, the Department of Electrical Engineering as a Research Associate. He later became an Assistant Professor, an Associate Professor, and in 2000 a Professor. In 2002, he moved to the Institute of Industrial Science, the University of Tokyo, as a Professor.

His research fields are control theory and its industrial application to motion control, electric vehicle, and welfare system, etc. He is now a Vice President of IEE-Japan IAS.