Analysis of Actuator Redundancy Resolution Methods for Bi-articularly Actuated Robot Arms

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Abstract—Bi-articular actuators — actuators that span two joints — are rising interest in robot application because they increase stability, optimize force production, and reduce the non-linearity of the end effector force as a function of force direction. In this paper, we propose an approach to resolve actuator redundancy for bi-articularly actuated robot arms in which the three actuators produce maximum joint actuator torques that differs among each other. A closed form solution based on the infinity norm is derived. The proposed infinity norm based approach is compared with the conventional 1-norm and 2-norm based methods. Under the same actuator limitations, the maximum end effector force produced with the proposed method is significantly greater than the one produced by the conventional methods. The proposed closed form solution is suitable for redundant systems with three inputs and two outputs, bringing the advantage of an higher maximum output without the need for iterative algorithms.

I. INTRODUCTION

Robot arms presenting animal musculo-skeletal characteristics such as bi-articular actuators — actuators that span two joints — have been proposed for more than two decades [1]. In recent years there has been increasing attention on such animal inspired robot arms, both in hardware and control design aspects. Regarding the hardware design, bi-articularly actuated robots have been realized by means of pneumatic actuators [2], [3], and motors with transmissions systems based on pulleys [4], [5], planetary gears [6], [7], wires [8], [9], and passive springs [10], [11].

All these robots are driven by more actuators than joints, resulting therefore in actuator redundancy. In order to resolve the actuator redundancy problem due to the presence of bi-articular actuators, many approaches have been proposed.

In [12] several animal inspired approaches such as fatigue minimization, muscle force minimization, total muscle metabolic energy consumption, total muscle stress minimization are compared among each others. Among these approaches, the muscle force minimization [13] is implemented on robot applications using the 1-norm.

Approaches based on pseudo-inverse matrices are used in the control design for kinematically redundant robot arm [14], [15]. Pseudo-inverse matrices are also used for actuator redundancy resolution [5], [16]. Moore-Penrose is the simplest pseudo-inverse matrix, and correspond to the minimization of the euclidean norm (2-norm) [17].

Iterative algorithms based on $\infty - norm$ optimization criteria have been used to resolve redundancy in kinematically redundant robot arms [18], [19]. In this paper the $\infty - norm$ is used to resolve the actuator redundancy for bi-articularly actuated robot arms. Differently from our previous work [20], the here proposed $\infty - norm$ based approach is extended also to the case in which the three actuators produce different maximum joint actuator torque among each other. A closed form solution based on a piecewise linear function for the infinity norm approach is proposed. The $\infty - norm$ approach allows to minimize the necessary maximum torque at each joint for a given force at end effector. Therefore it is an approach to optimize actuators design for robot arm equipped with bi-articular actuators.

In addition, the proposed $\infty - norm$ is analytically compared with the traditional 1-norm and 2-norm approaches both in terms of joint actuator input torques and maximum output force at the end effector.

In Section II main features and statics of robot arms equipped with bi-articular actuators are described. Then, in Section III, three approaches for torque distribution resolution — minimization of muscle force (1-norm), 2-norm and $\infty - norm$ — are introduced. In Section IV the characteristics of 1-norm, 2-norm and $\infty - norm$ approaches are analyzed in terms of joint actuator input torques and maximum output force at the end effector. Finally, in Section V, the advantages of the proposed optimization criteria are summarized.

II. CHARACTERISTICS AND MODELING OF ROBOT ARM WITH BI-ARTICULAR ACTUATORS

In conventional robot arms each joint is driven by one actuator. In the contrary, animal limbs present a complex musculoskeletal structure based on two types of muscles:

1) Monoarticular: the contraction of one of these muscles produces a torque on one joint.

2) Bi-articular: the contraction of one of these muscles produces the same torque on two consecutive joints at the same time. Gastrocnemius is an example of bi-articular muscle in the human leg.

A simplified model of the complex animal musculoskeletal system is shown in Fig. 1. This model is based on 6 contractile
actuators — extensors (e1, e2 and e3) and flexors (f1, f2 and f3) — coupled in three antagonistic pairs:

- e1–f1 and e2–f2: couples of mono-articular actuators that produce torques about joint 1 and 2, respectively.
- e3–f3: couple of bi-articular actuators that produce torque about joint 1 and 2 contemporaneously.

Robot arms driven by bi-articular actuators have numerous advantages: dramatical increase in range of end effector impedance which can be achieved without feedback [1], realization of path tracking and disturbance rejection using just feedforward control [21], improvement of balance control for jumping robots that do not use force sensors [22]. Moreover, multi-joints actuators such as tri-articular actuators, increase the efficiency in the output force for robot arm [4]. Another advantage of robot arm equipped with bi-articular actuators is the ability to produce a more homogeneously distributed maximum output force at the end effector [20], [23].

III. ACTUATOR REDUNDANCY PROBLEM AND RESOLUTION METHODS

The resulting statics of the bi-articularly actuated arm of Fig. 1 are shown in Fig. 2, where:

\[
T_1 = \tau_1 + \tau_3 \tag{1}
\]

\[
T_2 = \tau_2 + \tau_3 \tag{2}
\]

- The total torques about joint 1 and 2 are \(T_1\) and \(T_2\), respectively.
- The torques produced by mono-articular actuators about joints 1 and 2 are \(\tau_1\) and \(\tau_2\), respectively. They are calculated from the actuator input forces \(e_i\) and \(f_i\) for \(i = (1, 2)\) as:

\[
\tau_1 = (f_1 - e_1)r \tag{3}
\]

\[
\tau_2 = (f_2 - e_2)r \tag{4}
\]

where \(r\) is the distance between the joint axis and the point where the muscle force is applied, consider to be the same for all the muscles and all joint angles.

- The bi-articular torque produced about both joints is \(\tau_3\):

\[
\tau_3 = (f_3 - e_3)r \tag{5}
\]

- \(\mathbf{F}\) is a general force at the end effector with magnitude \(F\) and direction \(\theta_f\).

A two-link robot arm with the statics shown in Fig. 2 presents actuator redundancy. Given \(\tau_1\), \(\tau_2\), and \(\tau_3\), it is possible to determine \(\mathbf{T}\), and therefore \(\mathbf{F}\) by using the Jacobian:

\[
\begin{bmatrix}
T_1 \\
T_2 \\
\end{bmatrix} = J \begin{bmatrix}
F_x \\
F_y \\
\end{bmatrix} \tag{6}
\]

where

\[
J = \begin{bmatrix}
-l_1\sin(\theta_1) - l_2\sin(\theta_1 + \theta_2) & -l_2\sin(\theta_1 + \theta_2) \\
l_1\cos(\theta_1) + l_2\cos(\theta_1 + \theta_2) & l_2\cos(\theta_1 + \theta_2)
\end{bmatrix} \tag{7}
\]

and \(F_x\) and \(F_y\) are the orthogonal projection of \(\mathbf{F}\) on the x-axis and y-axis, respectively.

On the other hand, given \(\mathbf{F}\), and therefore \(\mathbf{T}\), it is generally not possible to determine uniquely \(\tau_1\), \(\tau_2\), and \(\tau_3\) (see (1) and (2)) due to the actuator redundancy. The problem represented by (1) and (2) is referred in the following as the redundancy actuator problem.

A. 1−norm approach

The actuator redundancy is resolved using the 1−norm by solving the following problem:

\[
\begin{align*}
\min & \quad \left( |\tau_1| + |\tau_2| + |\tau_3| \right) \\
\text{s.t.} & \quad T_1 = \tau_1 + \tau_3 \\
& \quad T_2 = \tau_2 + \tau_3 \tag{8}
\end{align*}
\]

where \(\tau_i^m, i = (1, 2, 3)\) is the maximum joint actuator torque that the actuator \(i\) can produce. The problem is solved using an iterative algorithm. Software tools as MATLAB can solve such problems.

Fig. 1. Two-link arm with four mono- and two bi-articular muslces: model and resulting forces at the end effector

Fig. 2. Statics of two-link arm with four mono- and two bi-articular actuators
B. 2-norm based approach

The actuator redundancy is resolved using the $2-$norm by solving the following problem:

$$\min \sqrt{\frac{(\tau_1)^2}{\tau_1^2} + \frac{(\tau_2)^2}{\tau_2^2} + \frac{(\tau_3)^2}{\tau_3^2}}$$

s.t. $T_1 = \tau_1 + \tau_3$

$$T_2 = \tau_2 + \tau_3$$  \hspace{1cm} (9)

The solution of the problem expressed by (9) is:

$$\begin{align*}
\tau_1 &= \frac{(T_2 - T_3)(\tau_1^m)^2}{(\tau_1^m)^2 + (\tau_2^m)^2 + (\tau_3^m)^2} + \frac{(T_1)(\tau_1^m)^2}{(\tau_1^m)^2 + (\tau_2^m)^2 + (\tau_3^m)^2} \\
\tau_2 &= \frac{2(T_2(\tau_1^m)^2 + (T_1 - T_3)(\tau_1^m)^2)(\tau_2^m)^2}{(\tau_1^m)^2 + (\tau_2^m)^2 + (\tau_3^m)^2} + \frac{(T_1)(\tau_2^m)^2}{(\tau_1^m)^2 + (\tau_2^m)^2 + (\tau_3^m)^2} \\
\tau_3 &= \frac{(T_1(\tau_1^m)^2)(\tau_3^m)^2}{(\tau_1^m)^2 + (\tau_2^m)^2 + (\tau_3^m)^2} + \frac{2(T_1)(\tau_3^m)^2}{(\tau_1^m)^2 + (\tau_2^m)^2 + (\tau_3^m)^2} \\
\end{align*}$$  \hspace{1cm} (10, 11, 12)

Proof of (10), (11), and (12) is reported in Appendix A.

If $\tau_1^m = \tau_2^m = \tau_3^m$ the solution becomes [20]:

$$\begin{align*}
\tau_1 &= \frac{2}{3}T_1 - \frac{1}{3}T_2 \\
\tau_2 &= -\frac{1}{3}T_1 + \frac{2}{3}T_2 \\
\tau_3 &= \frac{1}{3}T_1 + \frac{1}{3}T_2
\end{align*}$$  \hspace{1cm} (13, 14, 15)

C. Infinity norm based approach

The actuator redundancy is resolved using the $\infty-$norm by solving the following problem:

$$\min \max \left( \frac{|\tau_1|}{\tau_1^m}, \frac{|\tau_2|}{\tau_2^m}, \frac{|\tau_3|}{\tau_3^m} \right)$$

s.t. $T_1 = \tau_1 + \tau_3$

$$T_2 = \tau_2 + \tau_3$$  \hspace{1cm} (16)

The fact that three torque values are scaled by the respective maximum torque guarantees that the solution, when exists, does not violate any of the three constraints:

$$\begin{align*}
-\tau_1^m &\leq \tau_1 \leq \tau_1^m \\
-\tau_2^m &\leq \tau_2 \leq \tau_2^m \\
-\tau_3^m &\leq \tau_3 \leq \tau_3^m
\end{align*}$$  \hspace{1cm} (17, 18, 19)

Let us define:

$$\begin{align*}
c_1 &= \frac{\tau_1^m - \tau_1^m}{\tau_1^m + \tau_2^m} \\
c_2 &= \frac{\tau_2^m + \tau_2^m}{\tau_1^m + \tau_2^m} \\
c_3 &= \frac{\tau_3^m - \tau_3^m}{\tau_1^m + \tau_3^m}
\end{align*}$$  \hspace{1cm} (20, 21, 22)

The three parameters $c_1$, $c_2$, and $c_3$ are defined for any maximum joint actuator torque, and are constant. A closed form solution of the problem (16) is determined on the basis of the values of $T_1$ and $T_2$ as follows:

$$\begin{align*}
\tau_1 &= \begin{cases}
\frac{T_1 - T_2}{2} & \text{if } T_1 \leq T_2 \\
\frac{T_1}{2} & \text{if } T_1 > T_2 \text{ and } |T_1| \leq |T_2| \\
0 & \text{if } T_1 > T_2 \text{ and } |T_1| > |T_2|
\end{cases} \\
\tau_2 &= \begin{cases}
\frac{T_2 - T_1}{2} & \text{if } T_1 \leq T_2 \\
\frac{T_2}{2} & \text{if } T_1 > T_2 \text{ and } |T_1| \leq |T_2| \\
0 & \text{if } T_1 > T_2 \text{ and } |T_1| > |T_2|
\end{cases} \\
\tau_3 &= \begin{cases}
\frac{T_3}{2} & \text{if } T_1 \leq T_2 \\
\frac{T_3}{2} & \text{if } T_1 > T_2 \text{ and } |T_1| \leq |T_2| \\
0 & \text{if } T_1 > T_2 \text{ and } |T_1| > |T_2|
\end{cases}
\end{align*}$$  \hspace{1cm} (23, 24, 25)

where

$$\begin{align*}
case_1 &= (T_1 \leq c_1 T_2 \text{ and } T_2 \geq c_3 T_1) \\
&\quad \text{or } (T_1 > c_1 T_2 \text{ and } T_2 < c_3 T_1) \\
case_2 &= (T_1 \geq c_1 T_2 \text{ and } T_2 \geq c_3 T_1) \\
&\quad \text{or } (T_1 < c_1 T_2 \text{ and } T_2 < c_3 T_1) \\
case_3 &= (T_2 \leq c_2 T_1 \text{ and } T_2 \geq c_3 T_1) \\
&\quad \text{or } (T_2 > c_2 T_1 \text{ and } T_2 < c_3 T_1)
\end{align*}$$

Proof of (23), (24), and (25) is reported in Appendix B.

It is trivial to verify that the three linear piecewise functions (23), (24), and (25) are continuous in all the domain $D = (T_1, T_2)$.

In summary, the values of $\tau_1$, $\tau_2$, and $\tau_3$ that produce a given $F$ at the end effector, are determined as follows:

1) Calculate the desired joint torques $T = J^T F$.

2) According to calculated $T_1$ and $T_2$, the three desired joint actuator torques are directly determined using the three piecewise linear function (23), (24), and (25).

When all the actuators produce the same maximum joint actuator torque, that is $\tau_1^m = \tau_2^m = \tau_3^m$, $c_1 = c_3 = 0$ and $c_2 = 1$, and the solution becomes as in the following [20]:

$$\begin{align*}
\tau_1 &= \begin{cases}
\frac{T_1 - T_2}{2} & \text{if } T_1 T_2 \leq 0 \\
\frac{T_1}{2} & \text{if } T_1 T_2 > 0 \text{ and } |T_1| \leq |T_2| \\
\frac{T_2}{2} & \text{if } T_1 T_2 > 0 \text{ and } |T_1| > |T_2|
\end{cases} \\
\tau_2 &= \begin{cases}
\frac{T_2 - T_1}{2} & \text{if } T_1 T_2 \leq 0 \\
\frac{T_2}{2} & \text{if } T_1 T_2 > 0 \text{ and } |T_1| \leq |T_2| \\
\frac{T_1}{2} & \text{if } T_1 T_2 > 0 \text{ and } |T_1| > |T_2|
\end{cases} \\
\tau_3 &= \begin{cases}
\frac{T_3}{2} & \text{if } T_1 T_2 \leq 0 \\
\frac{T_3}{2} & \text{if } T_1 T_2 > 0 \text{ and } |T_1| \leq |T_2| \\
\frac{T_2}{2} & \text{if } T_1 T_2 > 0 \text{ and } |T_1| > |T_2|
\end{cases}
\end{align*}$$  \hspace{1cm} (26, 27, 28)

IV. Results

In the following a two-links robot arm with $l_1 = 15 m$ and $l_2 = 12 m$ is taken into account. The maximum joint actuator torques are $\tau_1^m = 16 N$, $\tau_2^m = 14 N$, and $\tau_3^m = 18 N$.
A. Joint actuator torques comparison

Fig. 3 shows a comparison of actuator joint torques input for three actuator redundancy resolution methods, $1 - norm$, $2 - norm$ and $\infty - norm$. The desired output is an output force $\mathbf{F}$ at the end effector with magnitude $1 \text{ N}$ in all the directions for $\theta_2 = 90^\circ$. The compared values are $\frac{|\tau_1|}{\sqrt{\tau_1^2+\tau_2^2+\tau_3^2}}$ (Fig. 3(b)), $\sqrt{\frac{\tau_1^2}{\tau_1^2} + \frac{\tau_2^2}{\tau_2^2} + \frac{\tau_3^2}{\tau_3^2}}$ (Fig. 3(c)), and $\max\left(\frac{|\tau_1|}{\tau_1^2}, \frac{|\tau_2|}{\tau_2^2}, \frac{|\tau_3|}{\tau_3^2}\right)$ (Fig. 3(d)).

The advantage of $1 - norm$ approach is that requires the lowest sum of actuator joint torques (Fig. 3(b)). The advantage of $\infty - norm$ is that allows to minimize the maximum actuator joint torques (Fig. 3(d)). $2 - norm$ is in between the other two approaches. The same results can be obtained for $\theta_2 \neq 90^\circ$.

B. Maximum output force

Fig. 4 shows the maximum output force at end effector for $\theta_2 = (30, 60, 90, 120, 150^\circ)$. Given the same maximum actuator joint torques $\infty - norm$ can produce the greatest force at end effector. This is the great advantage of the $\infty - norm$ approach. $1 - norm$ produces the lower output force, while $2 - norm$ is in between the other two approaches.

V. Conclusions

In this paper, we propose an approach to resolve actuator redundancy for bi-articularly actuated robot arms in which the three actuators produce maximum joint actuator torques that differs among each other. Moreover, the proposed $\infty - norm$ based approach is compared with the muscle force minimization approach ($1 - norm$) and with the Moore-Penrose pseudoinverse matrix ($2 - norm$) approaches. Main results are:

- The proposed $\infty - norm$ approach allows the maximization of output force at the end effector.
- The $1 - norm$ approach minimizes the total muscle force input, but the maximum force space is the smallest. If a circular output force the end effector is desired, this approach requires the highest joint actuator torque.
- The $2 - norm$ approach is in between $1 - norm$ and $\infty - norm$ in terms of both maximum force space and total torque input.
ACKNOWLEDGMENT

This work is supported by Inamori Foundation

APPENDIX A

PROOF OF CLOSED FORM SOLUTION FOR THE 2-NORM APPROACH

The problem expressed by (9) is written for a simpler notation as:

\[
\begin{align*}
\min & \quad \sqrt{\frac{(x)^2}{(mx)^2} + \frac{(y)^2}{(my)^2} + \frac{(z)^2}{(mz)^2}} \\
\text{s.t.} & \quad T_1 = x + z, \\
& \quad T_2 = y + z
\end{align*}
\]  
(29)

where \( T_1 \) and \( T_2 \) are the desired joint torques (known), \( x, y, \) and \( z \) are the desired joint actuator torques \( \tau_1, \tau_2, \) and \( \tau_3 \) (unknown), respectively; \( mx = \tau_1^m, \) \( my = \tau_2^m, \) and \( mz = \tau_3^m. \) Taking into account \( \mathbb{R}^3, \) the solution \((x, y, z)\) which satisfy \( \sqrt{\frac{(x)^2}{(mx)^2} + \frac{(y)^2}{(my)^2} + \frac{(z)^2}{(mz)^2}} \) has to meet the following three requirements:

1) To be on the line defined by

\[
T_1 = x + z
\]  
(30)

\[
T_2 = y + z
\]  
(31)

2) To be on the ellipsoid surface defined by:

\[
\frac{x^2}{mx^2} + \frac{y^2}{my^2} + \frac{z^2}{mz^2} = k
\]  
(32)

where \( k \) is a constant.

3) The plane passing through the line defined by (30) and (31) has to be tangent to the ellipsoid defined by (32). Hence:

\[
\frac{1}{mx^2} \frac{\partial^2}{\partial x^2} + \frac{1}{my^2} \frac{\partial^2}{\partial y^2} + \frac{1}{mz^2} \frac{\partial^2}{\partial z^2} = 0
\]  
(33)

Combining (30), (31), (32), (33) straightforward follows the solution of the problem (29):

\[
\begin{align*}
x &= (T_1 - T_2)mx^2 + T_1mx^2y^2 \\
y &= T_2mx^2 + (T_2 - T_1)my^2z^2 \\
z &= T_1my^2z^2 + T_2my^2z^2
\end{align*}
\]  
(34)

\[
\begin{align*}
x &= T_1myz^2 + T_2myz^2 \\
y &= T_1mz^2 + T_2mz^2 \\
z &= T_1mxz^2 + T_2mxz^2
\end{align*}
\]  
(35)

Equations (34), (35), and (36) correspond to (10), (11), and (12), respectively.

APPENDIX B

PROOF OF CLOSED FORM SOLUTION FOR THE INFINITY-NORM APPROACH

The problem expressed by (16) is written for a simpler notation as:

\[
\begin{align*}
\min & \quad \max \left( \frac{|x|}{mx}, \frac{|y|}{my}, \frac{|z|}{mz} \right) \\
\text{s.t.} & \quad T_1 = x + z, \\
& \quad T_2 = y + z
\end{align*}
\]  
(37)

where \( T_1 \) and \( T_2 \) are the desired joint torques (known), \( x, y, \) and \( z \) are the desired joint actuator torques \( \tau_1, \tau_2, \) and \( \tau_3 \) (unknown), respectively; \( mx = \tau_1^m, \) \( my = \tau_2^m, \) and \( mz = \tau_3^m. \)

A closed form solution of (37) is determined in the following. The searched solution has to satisfy at least one of the three equations \( \frac{|x|}{mx} = \frac{|y|}{my}, \frac{|y|}{my} = \frac{|z|}{mz}, \frac{|z|}{mz} = \frac{|x|}{mx}. \) In fact, when one of three variable’s absolute value decreases at least one of the other two increases. Therefore for any solution of the system with \( \frac{|x|}{mx} \neq \frac{|y|}{my} \neq \frac{|z|}{mz} \) it is possible to decrease the higher value among the three so to be equal to at least one of the other two. Therefore the searched solution is one among the following six:

1) \( \frac{x}{mx} = -\frac{y}{my} \)

\[
\begin{align*}
x + z &= T_1 \\
y + z &= T_2 \\
\frac{x}{mx} &= -\frac{y}{my}
\end{align*}
\]  
(38)

2) \( \frac{y}{my} = \frac{z}{mz} \)

\[
\begin{align*}
x + z &= T_1 \\
y + z &= T_2 \\
\frac{x}{mx} &= \frac{z}{mz}
\end{align*}
\]  
(39)

3) \( \frac{x}{mx} = \frac{z}{mz} \)

\[
\begin{align*}
x + z &= T_1 \\
y + z &= T_2 \\
\frac{x}{mx} &= \frac{y}{my}
\end{align*}
\]  
(40)

4) \( \frac{x}{mx} = \frac{y}{my} \)

\[
\begin{align*}
x + z &= T_1 \\
y + z &= T_2 \\
\frac{x}{mx} &= \frac{y}{my}
\end{align*}
\]  
(41)

5) \( \frac{y}{my} = -\frac{z}{mz} \)

\[
\begin{align*}
x + z &= T_1 \\
y + z &= T_2 \\
\frac{x}{mx} &= -\frac{z}{mz}
\end{align*}
\]  
(42)

6) \( \frac{x}{mx} = -\frac{z}{mz} \)

\[
\begin{align*}
x + z &= T_1 \\
y + z &= T_2 \\
\frac{x}{mx} &= -\frac{z}{mz}
\end{align*}
\]  
(43)

Let us define:

\[
\begin{align*}
c_1 &= \frac{mz - mx}{mz + my} \\
c_2 &= \frac{mx + my}{mz + mx} \\
c_3 &= \frac{mz - my}{mz + mx}
\end{align*}
\]  
(44)

These values depend only on the hardware characteristics of the arm, therefore are constant. Among the six possible solutions the searched one is directly selected on the basis
of $T_1$ and $T_2$ as follows (the variable subscript represents the respective equation number):

- If $(T_1 \leq c_1 T_2$ and $T_2 \geq c_2 T_1)$ or $(T_1 > c_1 T_2$ and $T_2 < c_2 T_1)$:

$$\left| \frac{x(38)}{mx} \right| = \left| \frac{y(38)}{my} \right| \geq \left| \frac{z(38)}{mz} \right|$$

$$\left| x(38) \right| \leq \left| x(39) \right| \quad \left| y(38) \right| \leq \left| y(40) \right| \quad \left| x(38) \right| \leq \left| x(41) \right| \quad \text{if } mz \geq my$$

Therefore solution is (38). In this case, $t_1$ in (23), $t_2$ in (24), and $t_3$ in (25), are equal to $x$ in (38), $y$ in (38), and $z$ in (38), respectively.

- If $(T_1 \geq c_1 T_2$ and $T_2 \geq c_2 T_1)$ or $(T_1 < c_1 T_2$ and $T_2 < c_2 T_1)$:

$$\left| \frac{y(39)}{my} \right| = \left| \frac{z(39)}{mz} \right| \geq \left| \frac{x(39)}{mx} \right|$$

$$\left| y(39) \right| \leq \left| y(40) \right| \quad \left| y(39) \right| \leq \left| y(41) \right| \quad \text{if } my \geq mx$$

Therefore solution is (39). In this case, $t_1$ in (23), $t_2$ in (24), and $t_3$ in (25), are equal to $x$ in (39), $y$ in (39), and $z$ in (39), respectively.

- If $(T_2 \leq c_2 T_1$ and $T_2 \geq c_2 T_1)$ or $(T_2 > c_2 T_1$ and $T_2 < c_2 T_1)$:

$$\left| \frac{x(40)}{mx} \right| = \left| \frac{z(40)}{mz} \right| \geq \left| \frac{y(40)}{my} \right|$$

$$\left| x(40) \right| \leq \left| x(38) \right| \quad \left| x(40) \right| \leq \left| x(39) \right| \quad \left| x(40) \right| \leq \left| x(41) \right| \quad \text{if } mx \geq my$$

Therefore solution is (40). In this case, $t_1$ in (23), $t_2$ in (24), and $t_3$ in (25), are equal to $x$ in (40), $y$ in (40), and $z$ in (40), respectively.

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