Generation Method of Admissible Sets for Mode Switching Control Using Final-State Control with Thrust Limitation

Yuma Yazaki*, Hiroshi Fujimoto**
The University of Tokyo
5-1-5, Kashiwanoha, Kashiwa, Chiba, 227-8561 Japan
Phone: +81-4-7136-3881*, +81-4-7136-4131**
Fax: +81-4-7136-3881*, +81-4-7136-4132**
Email: yazaki@hflab.k.u-tokyo.ac.jp*, fujimoto@k.u-tokyo.ac.jp**

***Abstract*—Mode Switching Control (MSC) is a control method which switches from one control mode to another according to switching conditions. MSC has been widely used in robots and hard disk drives. Considering thrust limitations, however, the switching condition is generally conservative since there are not any clear criteria to switch the control modes. Therefore, this paper proposes a new index of switching condition which uses initial state variables in Final-State Control (FSC) with thrust limitations. The novelty of the index is that feedforward inputs which take an initial state to a final state in finite time can be generated automatically, while taking into account thrust limitations. The effectiveness of the proposed index is shown by simulations.**

***Index Terms*—Final-State Control, Mode-Switching Control, Thrust Limitation, Initial State Variable, Energy Minimization, Jerk Minimization**

I. Introduction

Mode Switching Control (MSC) is a control method which switches from one control mode to another according to switching conditions. MSC has been widely used in robots and hard disk drives which need to switch between several types of control while in motion. Previous studies of MSC have been mainly performed about the following two themes; (i) to set initial state variables in a controller properly at the time of switching control modes and (ii) to design a switching condition to improve the transient response.

In a previous study on the former theme, Yamaguchi et al proposed an initial state value compensation which calculates the initial state of the controller by multiplying a real coefficient matrix designed in advance [1]. In another previous study on the former theme, an initial state value compensation by using feedforward (FF) input is proposed [2]. That is because setting initial state variables in a controller properly is equal to add a FF input to the output of the controller.

On the other hand, in a study on the latter theme, a method to determine the optimal initial state variables from some cost functions was proposed, and its effectiveness was demonstrated by experiments [3].

However, the index to switch control modes is only based on the tracking error in most of studies on the MSC and the switching condition is decided by the experience of the designer because there are no clear criteria for switching control modes. For this reason, generally a conservative design is preferred in order not to saturate thrust of the actuator. Therefore, this paper proposes a new index which enables to evaluate switching conditions quantitatively, considering actuator thrust limitation. This criterion uses initial state variables in Final-State Control (FSC) [4] with thrust limitations.

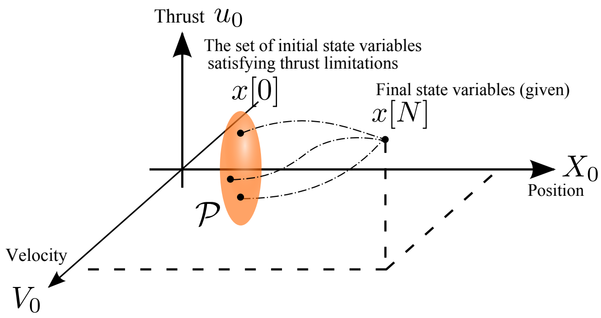


Fig. 1. Conceptual diagram of the set of initial state variables.

FSC is a control method which takes an initial state to a desired final state in finite time by applying FF inputs. FSC has been widely used in the short track-seeking control of hard disk drives, etc. There are notable studies derived from FSC: Frequency-shaped FSC (FFSC) [5], which designs FF input not to excite high resonance modes; Ploynominal-input-type FFSC (PFFSC) [6], which approximates the FF inputs as a polynomial; LMI PFFSC [7], which considers a limitation of the control input; Updating FSC (UFSC) [8], which can be applied to a time-varying final state. These previous researches on FSC assume that initial state variables are equal to zero. However, when FSC is applied to MSC, initial state variables are not zero because FSC in this case begins in the middle of an action.

The purpose of this study is to determine the set $\mathcal{P}$ of initial state variables $x[0]$ in FSC. The finite characteristics of the set is that initial state variables can be taken to a desired final state by using FSC while satisfying thrust limitations when the initial state variables is inside the set. Note that the thrust limitations and the final state variables $x[N]$ have to be given in advance to generate the set $\mathcal{P}$. Therefore, whether initial state variables can be taken to a desired final state variables can be judged by the set $\mathcal{P}$.

The outline of this paper is as follows. Section II derives the condition of the initial state variables in FSC with thrust limitations. The set $\mathcal{P}$ itself and its effectiveness are shown by simulation results in Section III. Finally, section IV presents the summary of the results and future works. Reference and appendix conclude this paper.

II. Derivation of the initial state variables conditions in FSC with thrust limitations

This section derives the conditions of the initial state variables in FSC with thrust limitations for the following two cases; (i) to minimize the energy cost and (ii) to minimize the jerk cost. In addition, the position, the velocity and the thrust are given as initial conditions in this paper.

A. Final State Control [4] [5]

FSC is a control method which takes an initial state to a final state in finite time by applying FF inputs. A state-space model of a discrete-time system is defined as follows:

$$\boldsymbol{x}[k+1] = \boldsymbol{A}\boldsymbol{x}[k] + \boldsymbol{B}u[k],\ y[k] \quad = \boldsymbol{C}\boldsymbol{x}[k]. \tag{1}$$

Let us consider the FF input $u[k]$ that takes an initial state $x[0]$ to a final state $x[N]$ in N steps. A performance index is set as follows:

$$J = \boldsymbol{U}^{\mathrm{T}}\boldsymbol{Q}\boldsymbol{U},\ \boldsymbol{Q} > 0,\ \boldsymbol{U} = [u[0]\ u[1]\ \cdots\ u[N-1]\,]^{\mathrm{T}}. \tag{2}$$

$\boldsymbol{Q} \in R^{N\times N}$ is a weighting matrix. The FF inputs minimizing (2) are given by

$$\boldsymbol{U} = \boldsymbol{Q}^{-1}\boldsymbol{\Sigma}^{\mathrm{T}}(\boldsymbol{\Sigma}\boldsymbol{Q}^{-1}\boldsymbol{\Sigma}^{\mathrm{T}})^{-1}(\boldsymbol{x}[N] - \boldsymbol{A}^{N}\boldsymbol{x}[0]), \tag{3}$$

$$\boldsymbol{\Sigma} = [\boldsymbol{A}^{N-1}\boldsymbol{B}\ \boldsymbol{A}^{N-2}\boldsymbol{B}\ \cdots\ \boldsymbol{B}]. \tag{4}$$

In this paper, u_i means $u[i]$. Limitations of the control input u_i is shown

$$-u_{\mathrm{lim}} \le u_i \le u_{\mathrm{lim}}. \tag{5}$$

Here, u_{lim} is the limitation of the control input. Section II-B and II-C describe the conditions of the initial state variables considering the limitations in (5).

Besides, a plant model $P(s)$ is defined as a rigid body model shown in (6) in this paper,

$$P(s) = \frac{1}{Ms^2}. \tag{6}$$

B. Condition of the initial state variables when defining the energy as a cost function

Control input minimizing energy consumption is required for applications whose control performance is degraded by the heat of the actuator. In this case, it is enough to derive feedforward input $\boldsymbol{U}$, because the dimension of $\boldsymbol{U}$ is force by applying FSC to (1). When the initial conditions of the position, the velocity and the thrust are given, (3) is transformed

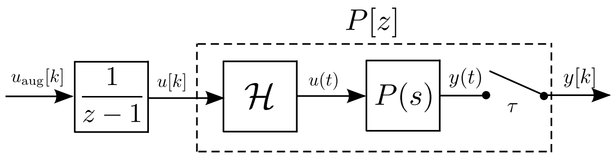


Fig. 2. Augmented system with an integrator.

into:

$$\boldsymbol{U}_{\mathrm{tr}} = \boldsymbol{Q}^{-1}\boldsymbol{\Sigma}_{\mathrm{tr}}^{\mathrm{T}}(\boldsymbol{\Sigma}_{\mathrm{tr}}\boldsymbol{Q}^{-1}\boldsymbol{\Sigma}_{\mathrm{tr}}^{\mathrm{T}})^{-1}\times (\boldsymbol{x}[N] - \boldsymbol{A}^{N}\boldsymbol{x}[0] - \boldsymbol{A}^{N-1}\boldsymbol{B}u[0]), \tag{7}$$

$$\boldsymbol{\Sigma}_{\mathrm{tr}} = [\boldsymbol{A}^{N-2}\boldsymbol{B}\ \cdots\ \boldsymbol{B}]. \tag{8}$$

By replacing each matrix with each element, (7) is rearranged as follows:

$$\begin{bmatrix} u_{\mathrm{tr}1} \\ u_{\mathrm{tr}2} \\ \vdots \\ u_{\mathrm{tr}N-1} \end{bmatrix} = \begin{bmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \\ \vdots & \vdots \\ q_{N-1\,1} & q_{N-1\,2} \end{bmatrix} \left\{ \begin{bmatrix} X_F \\ V_F \end{bmatrix} - \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} X_0 \\ V_0 \end{bmatrix} - \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} u_0 \right\}. \tag{9}$$

Here, $u_{\mathrm{tr}i}$ $(1 \le i \le N-1)$ is an element in the i-th row of $\boldsymbol{U}_{\mathrm{tr}}$, q_{ij} $(1 \le i \le N-1,\ 1 \le j \le 2)$ is an element in the i-th row and the j-th column of $\boldsymbol{Q}^{-1}\boldsymbol{\Sigma}_{\mathrm{tr}}^{\mathrm{T}}(\boldsymbol{\Sigma}_{\mathrm{tr}}\boldsymbol{Q}^{-1}\boldsymbol{\Sigma}_{\mathrm{tr}}^{\mathrm{T}})^{-1}$, a_{ij} $(1 \le i \le 2,\ 1 \le j \le 2)$ is an element in the i-th row and the j-th column of $\boldsymbol{A}^{N}$, b_i $(1 \le i \le 2)$ is an element in the i-th row of $\boldsymbol{A}^{N-1}\boldsymbol{B}$. $X_0,\ V_0,\ u_0$ are initial states of position, velocity and thrust while $X_F,\ V_F$ are final states of position and velocity, respectively.

The analysis solutions $u_{\mathrm{tr}i}$ must be derived to consider the thrust limitations. Hence, the analysis solutions of q_{ij} are needed and the solutions are obtained by calculating $\boldsymbol{Q}^{-1}\boldsymbol{\Sigma}_{\mathrm{tr}}^{\mathrm{T}}(\boldsymbol{\Sigma}_{\mathrm{tr}}\boldsymbol{Q}^{-1}\boldsymbol{\Sigma}_{\mathrm{tr}}^{\mathrm{T}})^{-1}$. In this section, $\boldsymbol{Q}$ is a unit matrix to minimize energy consumption. Derivations of each coefficient are shown in appendix A. By combining (5) and (9), the condition of initial state variables in FSC with thrust limitations is given as follows:

$$-u_{\mathrm{lim}} \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} \le \begin{bmatrix} u_0 \\ \vdots \\ u_{\mathrm{tr}N-1} \end{bmatrix} \le u_{\mathrm{lim}} \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}. \tag{10}$$

Each of the N rows in (10) expresses a plane, which has $X_0,\ V_0,\ u_0$ as variables. The set of solutions for the condition of initial state variables in FSC with thrust limitations is limited by these $2N$ planes in the Cartesian space $\{X_0, V_0, u_0\}$.

C. Condition of the initial state variables when defining jerk as a cost function

A smooth control input is required not to excite the mechanical vibration modes in the case of applications where resonance frequency inhibits the improvement of its control performance, such as hard disk drives. In addition, the method proposed in this section has the advantage of being able to designate the thrust of the final state while the method proposed in section II-B can not designate it. The method proposed in this section needs an augmented system with an integrator to evaluate the differential of the actual control input and to add the thrust element to state variables. The augmented system is shown in Fig. 2 and is defined as follows:

$$\boldsymbol{x}_{\text{aug}}[k+1] = \boldsymbol{A}_{\text{aug}}\boldsymbol{x}_{\text{aug}}[k] + \boldsymbol{B}_{\text{aug}}u_{\text{aug}}[k], \quad y[k] = \boldsymbol{C}_{\text{aug}}\boldsymbol{x}_{\text{aug}}[k]. \tag{11}$$

The FF inputs $\boldsymbol{U}_{\text{aug}}$ are given by

$$\boldsymbol{U}_{\text{aug}} = \boldsymbol{Q}^{-1}\boldsymbol{\Sigma}_{\text{aug}}^{\text{T}}(\boldsymbol{\Sigma}_{\text{aug}}\boldsymbol{Q}^{-1}\boldsymbol{\Sigma}_{\text{aug}}^{\text{T}})^{-1}(\boldsymbol{x}[N] - \boldsymbol{A}^{N}\boldsymbol{x}[0]), \tag{12}$$

$$\boldsymbol{\Sigma}_{\text{aug}} = [\boldsymbol{A}^{N-1}\boldsymbol{B} \ \cdots \ \boldsymbol{B}]. \tag{13}$$

Note that $u_{\text{aug}}[k]$ is an input whose dimension corresponds to jerk because of the augmented system shown in Fig. 2. By replacing each matrix with each element, (12) is rearranged as follows:

$$\begin{bmatrix} u_{\text{aug}1} \\ u_{\text{aug}2} \\ \vdots \\ u_{\text{aug}N} \end{bmatrix} = \begin{bmatrix} q_{11} & q_{12} & q_{13} \\ q_{21} & q_{22} & q_{23} \\ \vdots & \vdots & \vdots \\ q_{N1} & q_{N2} & q_{N3} \end{bmatrix} \left\{ \begin{bmatrix} X_F \\ V_F \\ u_F \end{bmatrix} - \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} X_0 \\ V_0 \\ u_0 \end{bmatrix} \right\}. \tag{14}$$

Here, $u_{\text{aug}i}$ $(1 \le i \le N)$ is an element in the i-th row of $\boldsymbol{U}$, q_{ij} $(1 \le i \le N,\ 1 \le j \le 3)$ is an element in the i-th row and the j-th column of $\boldsymbol{Q}^{-1}\boldsymbol{\Sigma}_{\text{aug}}^{\text{T}}(\boldsymbol{\Sigma}_{\text{aug}}\boldsymbol{Q}^{-1}\boldsymbol{\Sigma}_{\text{aug}}^{\text{T}})^{-1}$, a_{ij} $(1 \le i \le 3,\ 1 \le j \le 3)$ is an element in the i-th row and the j-th column of $\boldsymbol{A}_{\text{aug}}^{N}$. $X_0,\ V_0,\ u_0$ are initial states of position, velocity and thrust while $X_F,\ V_F,\ u_F$ are final states of position, velocity and thrust, respectively.

The analysis solutions $u_{\text{aug}i}$ must be derived to consider the thrust limitations. Hence, the analysis solutions of q_{ij} are needed and the solutions are obtained by calculating $\boldsymbol{Q}^{-1}\boldsymbol{\Sigma}_{\text{aug}}^{\text{T}}(\boldsymbol{\Sigma}_{\text{aug}}\boldsymbol{Q}^{-1}\boldsymbol{\Sigma}_{\text{aug}}^{\text{T}})^{-1}$. In this section, $\boldsymbol{Q}$ is an unit matrix to minimize the jerk. Derivations of each coefficient are shown in appendix B. Note that it is necessary to convert $u_{\text{aug}}[k]$ to u_i because the dimension of $u_{\text{aug}}[k]$ is not thrust but jerk. Therefore, u_i is given in (15) by using the initial thrust u_0 and $u_{\text{aug}i}$.

$$u_i = \begin{cases} u_0, & i = 0 \\ \displaystyle\sum_{k=1}^{i} u_{\text{aug}k} + u_0, & i \ge 1 \end{cases} \tag{15}$$

By combining (5) and (15), the condition of initial state variables in FSC with thrust limitations is given as follows:

$$-u_{\text{lim}} \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \\ 1 \end{bmatrix} \le \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} u_0 \\ u_{\text{aug}1} \\ \vdots \\ u_{\text{aug}N-1} \\ u_{\text{aug}N} \end{bmatrix} \le u_{\text{lim}} \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \\ 1 \end{bmatrix}. \tag{16}$$

Each of the $N+1$ rows in (16) expresses a plane, which has $X_0,\ V_0,\ u_0$ as variables. The set of solutions for the condition of initial state variables in FSC with thrust limitations is limited by these $2N+2$ planes in the Cartesian space $\{X_0, V_0, u_0\}$.

III. Simulation

This section shows the condition of initial state variables itself and its effectiveness by simulation results.

In this simulation, the mass of (6) is $M = 6\,\text{kg}$, the control period is $200\,\mu\text{s}$, the final state is set at $[X_N, V_N, u_N] = [0.1, 0.4, 0]$ and the limitation of the control input is $u_{\text{lim}} = 5.0\,\text{N}$.

TABLE I
CONDITIONS OF INITIAL STATE VARIABLES.

Case	Type of dots	Initial states $[X_0, V_0, u_0]$
1	Red asterisk	[0.0595, 0.37, 2.0]
2	Yellow triangle	[0.0620, 0.39, 2.0]
3	Black circle	[0.0600, 0.38, 2.0]
4	Green square	[0.0615, 0.38, 2.0]

A. Simulation 1

This first simulation draws the conditions of initial state variables themselves.

Plot ranges are $X_0 \in [0,\ 0.1]$, $V_0 \in [0,\ 0.4]$ and $u_0 \in [-5,\ 5]$ and lattice points are provided with proper spaces. Moreover, (10) or (16) is calculated to each lattice point and it is plotted when it satisfies all of the conditions. The conditions are calculated for the case of $N = 100,\ 300,\ 500$.

The condition in the case of minimizing the energy cost is shown in (a) and (b) of Fig. 3–5. The condition in the case of minimizing the jerk cost is shown in (c) and (d) of Fig. 3–5.

As shown in Fig. 3–5, the conditions of the initial state variables in each case are time-varying set, and the smaller the number of steps N becomes, the smaller the set is.

B. Simulation 2

This second simulation demonstrates the effectiveness of the proposed conditions by simulation results in time domain. Four kinds of initial state variables are prepared for this simulation and the states are shown in Table I. The initial state variables are located outside the proposed set in the Cases 1 and 2 and they are located inside in the Cases 3 and 4. Therefore, it is expected that the thrust derived from FSC will violate the thrust limitations in the Cases 1 and 2. On the other hand, it is expected that the thrust derived from FSC will not violate the thrust limitations in the Cases 3 and 4.

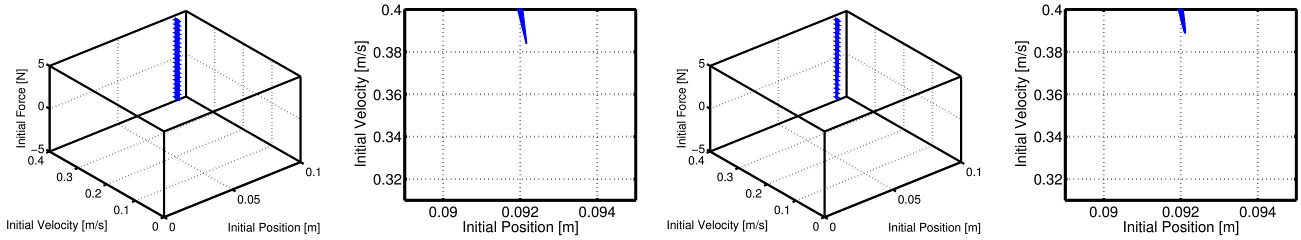
In this simulation, the thrust derived from FSC whose evaluation function is jerk is described. The number of steps is $N = 500$. Fig. 6(a) shows a 2D graph of Fig. 5(c) at $u_0 = 2.0\,\text{N}$.

The changes of state variables and the thrust derived from FSC are shown in Fig. 6(b), (c). As shown in Fig. 6(b), initial state variables can be taken to a desired final state variables by FSC in all cases. However, as shown in Fig. 6(c), it is found that the control input derived from FSC satisfies thrust limitations only in Case 3 and 4 whose initial state variables are inside the proposed set.

In this paper, note that the thrust derived from this method does not saturate in $t \in [0, N\tau]$ because (10) or (16) is considered after obtaining an analytical solution of QP problem (9) or (14) with the assumption that the initial states are unknown variables.

IV. Conclusion

This paper proposes a new index to switch control modes which uses conditions of the initial state variables in FSC with thrust limitations. The conditions need to apply FSC to MSC. Thus, the conditions are derived in the case of defining



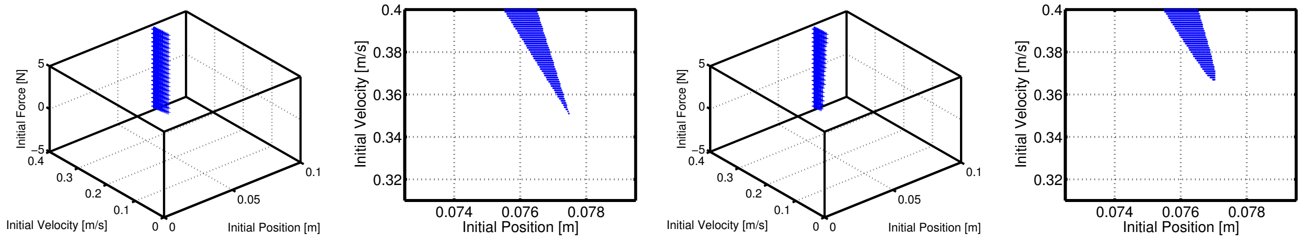
(a) Condition of initial state variables with energy minimization.

(b) 2D graph of Fig. 3(a) at $u_0 = 0$.

(c) Condition of initial state variables with jerk minimization.

(d) 2D graph of Fig. 3(c) at $u_0 = 0$.

Fig. 3. Condition of initial state variables ($N = 100$).



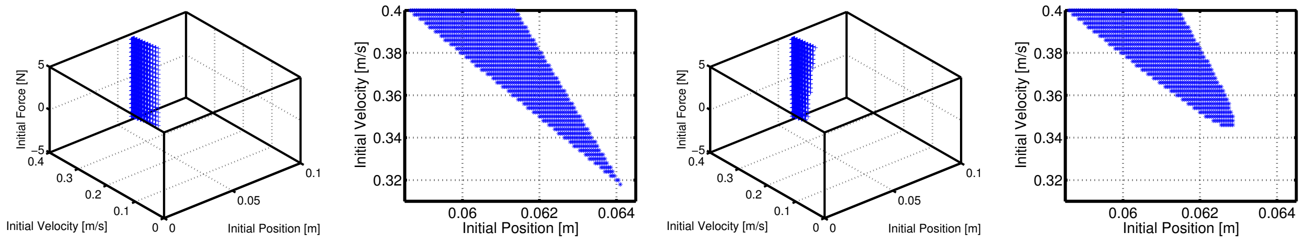
(a) Condition of initial state variables with energy minimization.

(b) 2D graph of Fig. 4(a) at $u_0 = 0$.

(c) Condition of initial state variables with jerk minimization.

(d) 2D graph of Fig. 4(c) at $u_0 = 0$.

Fig. 4. Condition of initial state variables ($N = 300$).



(a) Condition of initial state variables with energy minimization.

(b) 2D graph of Fig. 5(a) at $u_0 = 0$.

(c) Condition of initial state variables with jerk minimization.

(d) 2D graph of Fig. 5(c) at $u_0 = 0$.

Fig. 5. Condition of initial state variables ($N = 500$).

energy or jerk as the evaluation function. The case in which jerk is defined as the evaluation function has the advantage of being able to designate the thrust of the final state. Moreover, simulation results demonstrate that the conditions are time-varying sets and that the proposed conditions can efficiently determine the time of switching control modes.

In this study, the dimension of the condition is three because the plant is defined as a rigid body model. However, even when the plant model has higher dimensions and is a more complicated expression, it is expected that the conditions are calculated in the same way. Then, the condition will be a set surrounded by many hyperplanes, which are difficult to draw.

However, a real-time calculation of the proposed condition with many constraints is difficult. Thus, we will propose a new mounting method enabling real-time calculation of the conditions in future research.

References

[1] T. Yamaguchi, H. Soyama, H. Hosokawa, K. Tsuneta, H. Hirai, "Mode switching control design with initial value compensation and its application to head positioning control on magnetic disk drives," *IEEE Transactions on Magnetics*, vol. 43, pp. 65–73, 1996

[2] T. Yamaguchi, H. Soyama, H. Hosokawa, K. Tsuneta, H. Hirai, "Improvement of settling response of disk drive head positioning servo using mode switching control with initial value compensation," *IEEE Transactions on Industrial Electronics*, vol. 32, pp. 1767–1772, 1996

[3] T. Yamaguchi, H. Numasato, H. Hirai, "A mode-switching control for motion control and its application to disk drives: design of optimal mode-switching conditions," *IEEE/ASME Transactions on Mechatronics*, vol. 3, pp. 202–209, 1998

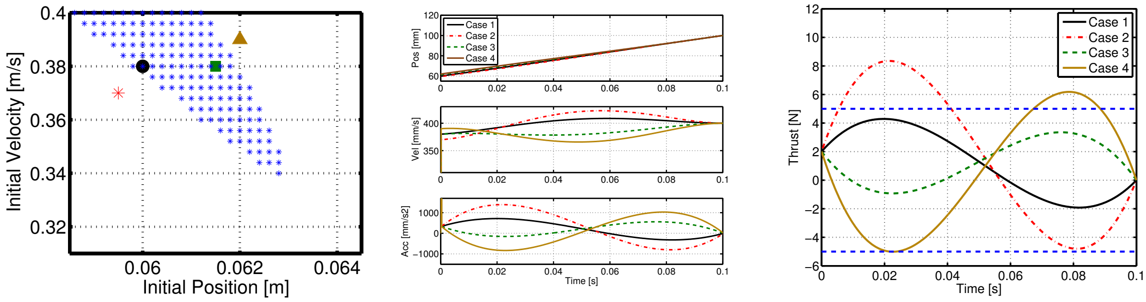
[4] T. Takayoshi, H. Nishimura, "Final-state control using compensation input," *Transactions of the Society of Instrument and Control Engineers*, vol. 30, pp. 253–260, 1994

[5] M. Hirata, T. Hasegawa, K. Nonami, "Short Track-seeking Control of Hard Disk Drives by Using Final-state Control," *IEEJ Trans Industry Applications*, vol. 125, pp. 524–529, 2005

[6] M. Hirata, F. Ueno, "Final-State Control Using Polynomial and Time-Series Data," *IEEE Transactions on Magnetics*, vol. 47, pp. 1944–1950, 2011

[7] M. Hirata, F. Ueno, "Polynomial-input-type final-state control taking account of input saturation," *Preprints of the 18th IFAC World Congress*, pp. 4061–4066, 2011

[8] T. Yoshiura, S. Hara, "Proposal of updating final-state control and its application to a connection control problem," *IEEE 13th International Workshop on Advanced Motion Control (AMC)*, pp. 681–686, 2014



(a) 2D graph of Fig. 5(c) at $u_0 = 2$.

(b) The change of position, velocity and acceleration.

(c) Thrust derived from FSC.

Fig. 6. Simulation results in time domain ($N = 500$).

APPENDIX A
DERIVATIONS OF COEFFICIENT IN (9)

Coefficients of a discrete-time state-space equation $\boldsymbol{A}$, $\boldsymbol{B}$ and $\boldsymbol{C}$ are given by (17) because the plant model $P(s)$ is defined in (6).

$$\boldsymbol{A} = \begin{bmatrix} 1 & \tau \\ 0 & 1 \end{bmatrix}, \quad \boldsymbol{B} = \begin{bmatrix} \frac{\tau^2}{2M} & \frac{\tau}{M} \end{bmatrix}^{\mathrm{T}}, \quad \boldsymbol{C} = \begin{bmatrix} 1 & 0 \end{bmatrix} \tag{17}$$

Here, τ is the control period. First of all, let us consider $\boldsymbol{Q}^{-1}\boldsymbol{\Sigma}^{\mathrm{T}}(\boldsymbol{\Sigma}\boldsymbol{Q}^{-1}\boldsymbol{\Sigma}^{\mathrm{T}})^{-1}$ in (3). $\boldsymbol{Q}$ is an unit matrix to minimize the energy consumption. $\boldsymbol{A}^n$, $\boldsymbol{A}^{n-1}\boldsymbol{B}$ need to be calculated because $\boldsymbol{\Sigma}$ is given in (8). a_{ij} and b_j are expressed in (18) by $\boldsymbol{A}^n$ and $\boldsymbol{A}^n\boldsymbol{B}$.

$$\boldsymbol{A}^n = \begin{bmatrix} 1 & n\tau \\ 0 & 1 \end{bmatrix}, \quad \boldsymbol{A}^n\boldsymbol{B} = \begin{bmatrix} 1 & n\tau \\ 0 & 1 \end{bmatrix}\begin{bmatrix} \frac{\tau^2}{2M} \\ \frac{\tau}{M} \end{bmatrix} = \begin{bmatrix} \frac{\tau^2}{M}\left\{\frac{1}{2}+n\right\} \\ \frac{\tau}{M} \end{bmatrix}, \qquad \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \boldsymbol{A}^k|_{k=N}, \quad \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \boldsymbol{A}^k\boldsymbol{B}|_{k=N-1} \tag{18}$$

Thus, $\boldsymbol{\Sigma}_{\mathrm{tr}}$ is given by (19)

$$\boldsymbol{\Sigma}_{\mathrm{tr}} = \begin{bmatrix} \boldsymbol{A}^{N-2}\boldsymbol{B} & \cdots & \boldsymbol{B} \end{bmatrix} = \begin{bmatrix} \frac{\tau^2}{M}\left\{\frac{1}{2}+(N-2)\right\} & \cdots & \frac{\tau^2}{M}\left\{\frac{1}{2}+(N-N)\right\} \\ \frac{\tau}{M} & \cdots & \frac{\tau}{M} \end{bmatrix} \tag{19}$$

Therefore, $\boldsymbol{\Sigma}_{\mathrm{tr}}\boldsymbol{\Sigma}_{\mathrm{tr}}^{\mathrm{T}}$ is provided in (20).

$$\boldsymbol{\Sigma}_{\mathrm{tr}}\boldsymbol{\Sigma}_{\mathrm{tr}}^{\mathrm{T}} = \begin{bmatrix} \frac{\tau^2}{M}\left\{\frac{1}{2}+(N-2)\right\} & \cdots & \frac{\tau^2}{M}\left\{\frac{1}{2}+(N-N)\right\} \\ \frac{\tau}{M} & \cdots & \frac{\tau}{M} \end{bmatrix}\begin{bmatrix} \frac{\tau^2}{M}\left\{\frac{1}{2}+(N-2)\right\} & \frac{\tau}{M} \\ \vdots & \vdots \\ \frac{\tau^2}{M}\left\{\frac{1}{2}+(N-N)\right\} & \frac{\tau}{M} \end{bmatrix} = \begin{bmatrix} \frac{\tau^4}{M^2}\sum_{k=0}^{N-2}\left(k+\frac{1}{2}\right)^2 & \frac{\tau^3}{M^2}\sum_{k=0}^{N-2}\left(k+\frac{1}{2}\right) \\ \frac{\tau^3}{M^2}\sum_{k=0}^{N-2}\left(k+\frac{1}{2}\right) & \frac{\tau^2}{M^2}(N-1) \end{bmatrix} \tag{20}$$

Next, a determinant $\left|\boldsymbol{\Sigma}_{\mathrm{tr}}\boldsymbol{\Sigma}_{\mathrm{tr}}^{\mathrm{T}}\right|$ and a cofactor matrix $\mathrm{adj}(\boldsymbol{\Sigma}_{\mathrm{tr}}\boldsymbol{\Sigma}_{\mathrm{tr}}^{\mathrm{T}})$ should be calculated in order to obtain the inverse matrix $(\boldsymbol{\Sigma}_{\mathrm{tr}}\boldsymbol{\Sigma}_{\mathrm{tr}}^{\mathrm{T}})^{-1}$. The determinant $\left|\boldsymbol{\Sigma}_{\mathrm{tr}}\boldsymbol{\Sigma}_{\mathrm{tr}}^{\mathrm{T}}\right|$ and the cofactor matrix $\mathrm{adj}(\boldsymbol{\Sigma}_{\mathrm{tr}}\boldsymbol{\Sigma}_{\mathrm{tr}}^{\mathrm{T}})$ are shown in (21), respectively.

$$\left|\boldsymbol{\Sigma}_{\mathrm{tr}}\boldsymbol{\Sigma}_{\mathrm{tr}}^{\mathrm{T}}\right| = \frac{\tau^6}{M^4}\left[\sum_{k=0}^{N-2}\left(k+\frac{1}{2}\right)^2(N+1) - \left\{\sum_{k=0}^{N-2}\left(k+\frac{1}{2}\right)\right\}^2\right], \quad \mathrm{adj}(\boldsymbol{\Sigma}_{\mathrm{tr}}\boldsymbol{\Sigma}_{\mathrm{tr}}^{\mathrm{T}}) = \begin{bmatrix} \frac{\tau^2}{M^2}(N-1) & -\frac{\tau^3}{M^2}\sum_{k=0}^{N-2}\left(k+\frac{1}{2}\right) \\ -\frac{\tau^3}{M^2}\sum_{k=0}^{N-2}\left(k+\frac{1}{2}\right) & \frac{\tau^4}{M}\sum_{k=0}^{N-2}\left(k+\frac{1}{2}\right)^2 \end{bmatrix} \tag{21}$$

Finally, $\boldsymbol{\Sigma}_{\mathrm{tr}}^{\mathrm{T}}\left(\boldsymbol{\Sigma}_{\mathrm{tr}}\boldsymbol{\Sigma}_{\mathrm{tr}}^{\mathrm{T}}\right)^{-1}$ is derived in (22).

$$\boldsymbol{\Sigma}_{\mathrm{tr}}^{\mathrm{T}}\left(\boldsymbol{\Sigma}_{\mathrm{tr}}\boldsymbol{\Sigma}_{\mathrm{tr}}^{\mathrm{T}}\right)^{-1} = \begin{bmatrix} \frac{\tau^2}{M}\left\{\frac{1}{2}+(N-2)\right\} & \frac{\tau}{M} \\ \vdots & \vdots \\ \frac{\tau^2}{M}\left\{\frac{1}{2}+(N-N)\right\} & \frac{\tau}{M} \end{bmatrix} \cdot \begin{bmatrix} \frac{\tau^2}{M^2}(N-1) & -\frac{\tau^3}{M^2}\sum_{k=0}^{N-2}\left(k+\frac{1}{2}\right) \\ -\frac{\tau^3}{M^2}\sum_{k=0}^{N-2}\left(k+\frac{1}{2}\right) & \frac{\tau^4}{M}\sum_{k=0}^{N-2}\left(k+\frac{1}{2}\right)^2 \end{bmatrix} \cdot \frac{M^4}{\tau^6}\frac{1}{\left[\sum_{k=0}^{N-2}\left(k+\frac{1}{2}\right)^2(N+1) - \left\{\sum_{k=0}^{N-2}\left(k+\frac{1}{2}\right)\right\}^2\right]} \tag{22}$$

Besides, the sum of the 1th, 2nd, 3rd and 4th power of a natural number is expressed by (23), respectively.

$$\sum_{k=1}^{n} k = \frac{1}{2}n(n+1), \quad \sum_{k=1}^{n} k^2 = \frac{1}{6}n(n+1)(2n+1), \quad \sum_{k=1}^{n} k^3 = \frac{1}{4}n^2(n+1)^2, \quad \sum_{k=1}^{n} k^4 = \frac{1}{30}n(n+1)(2n+1)(3n^2+3n-1) \tag{23}$$

By substituting (23) into (22), (24) is obtained.

$$\begin{aligned} q_{i1} &= \frac{6M}{\tau^2(N-2)(N-1)}(N-2i+2) \\ q_{i2} &= \frac{M}{\tau(N-2)(N-1)N}\left\{-2N^2+(-5+6i)N+(6-6i)\right\} \end{aligned} \tag{24}$$

B DERIVATIONS OF COEFFICIENT IN (14)

Coefficients of a state-space equation of an augment system $\boldsymbol{A}_{\mathrm{aug}}$, $\boldsymbol{B}_{\mathrm{aug}}$ and $\boldsymbol{C}_{\mathrm{aug}}$ are given by (25).

$$\boldsymbol{A}_{\mathrm{aug}}=\begin{bmatrix}1 & \tau & \frac{\tau^2}{2M}\\ 0 & 1 & \frac{\tau}{M}\\ 0 & 0 & 1\end{bmatrix},\ \boldsymbol{B}_{\mathrm{aug}}=\begin{bmatrix}0 & 0 & 1\end{bmatrix}^{\mathrm{T}},\ \boldsymbol{C}_{\mathrm{aug}}=\begin{bmatrix}1 & 0 & 0\end{bmatrix} \tag{25}$$

First of all, let us consider $\boldsymbol{Q}^{-1}\boldsymbol{\Sigma}_{\mathrm{aug}}^{\mathrm{T}}(\boldsymbol{\Sigma}_{\mathrm{aug}}\boldsymbol{Q}^{-1}\boldsymbol{\Sigma}_{\mathrm{aug}}^{\mathrm{T}})^{-1}$ in (12). $\boldsymbol{Q}$ is an unit matrix to minimize the jerk. $\boldsymbol{A}_{\mathrm{aug}}^{n}$ and $\boldsymbol{A}_{\mathrm{aug}}^{n}\boldsymbol{B}_{\mathrm{aug}}$ need to be calculated because $\boldsymbol{\Sigma}_{\mathrm{aug}}$ is given in (4). a_{ij} is expressed in (26) by $\boldsymbol{A}_{\mathrm{aug}}^{n}$ and $\boldsymbol{A}_{\mathrm{aug}}^{n}\boldsymbol{B}_{\mathrm{aug}}$.

$$\boldsymbol{A}_{\mathrm{aug}}^{n}=\begin{bmatrix}1 & n\tau & \frac{n^2\tau^2}{2M}\\ 0 & 1 & \frac{n\tau}{M}\\ 0 & 0 & 1\end{bmatrix},\ \boldsymbol{A}_{\mathrm{aug}}^{n}\boldsymbol{B}_{\mathrm{aug}}=\begin{bmatrix}1 & n\tau & \frac{n^2\tau^2}{2M}\\ 0 & 1 & \frac{n\tau}{M}\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}0\\0\\1\end{bmatrix}=\begin{bmatrix}\frac{n^2\tau^2}{2M}\\ \frac{n\tau}{M}\\ 1\end{bmatrix},\qquad \begin{bmatrix}a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ a_{31} & a_{32} & a_{33}\end{bmatrix}=\boldsymbol{A}_{\mathrm{aug}}^{k}|_{k=N} \tag{26}$$

Thus, $\boldsymbol{\Sigma}_{\mathrm{aug}}$ is given by (27)

$$\boldsymbol{\Sigma}_{\mathrm{aug}}=\begin{bmatrix}\boldsymbol{A}_{\mathrm{aug}}^{N-1}\boldsymbol{B}_{\mathrm{aug}} & \boldsymbol{A}_{\mathrm{aug}}^{N-2}\boldsymbol{B}_{\mathrm{aug}} & \cdots & \boldsymbol{B}_{\mathrm{aug}}\end{bmatrix}=\begin{bmatrix}\frac{(N-1)^2\tau^2}{2M} & \frac{(N-2)^2\tau^2}{2M} & \cdots & 0\\ \frac{(N-1)\tau}{M} & \frac{(N-2)\tau}{M} & \cdots & 0\\ 1 & 1 & \cdots & 1\end{bmatrix} \tag{27}$$

Therefore, $\boldsymbol{\Sigma}_{\mathrm{aug}}\boldsymbol{\Sigma}_{\mathrm{aug}}^{\mathrm{T}}$ is provided in (28).

$$\boldsymbol{\Sigma}_{\mathrm{aug}}\boldsymbol{\Sigma}_{\mathrm{aug}}^{\mathrm{T}}=\begin{bmatrix}\frac{(N-1)^2\tau^2}{2M} & \frac{(N-2)^2\tau^2}{2M} & \cdots & 0\\ \frac{(N-1)\tau}{M} & \frac{(N-2)\tau}{M} & \cdots & 0\\ 1 & 1 & \cdots & 1\end{bmatrix}\begin{bmatrix}\frac{(N-1)^2\tau^2}{2M} & \frac{(N-1)\tau}{M} & 1\\ \frac{(N-2)^2\tau^2}{2M} & \frac{(N-2)\tau}{M} & 1\\ \vdots & \vdots & \vdots\\ 0 & 0 & 1\end{bmatrix}=\begin{bmatrix}\frac{\tau^4}{4M^2}\sum_{k=0}^{N-1}k^4 & \frac{\tau^3}{2M^2}\sum_{k=0}^{N-1}k^3 & \frac{\tau^2}{2M}\sum_{k=0}^{N-1}k^2\\ \frac{\tau^3}{2M^2}\sum_{k=0}^{N-1}k^3 & \frac{\tau^2}{M^2}\sum_{k=0}^{N-1}k^2 & \frac{\tau}{M}\sum_{k=0}^{N-1}k\\ \frac{\tau^2}{2M}\sum_{k=0}^{N-1}k^2 & \frac{\tau}{M}\sum_{k=0}^{N-1}k & N\end{bmatrix} \tag{28}$$

Next, a determinant $\left|\boldsymbol{\Sigma}_{\mathrm{aug}}\boldsymbol{\Sigma}_{\mathrm{aug}}^{\mathrm{T}}\right|$ and a cofactor matrix $\mathrm{adj}(\boldsymbol{\Sigma}_{\mathrm{aug}}\boldsymbol{\Sigma}_{\mathrm{aug}}^{\mathrm{T}})$ should be calculated in order to obtain the inverse matrix $(\boldsymbol{\Sigma}_{\mathrm{aug}}\boldsymbol{\Sigma}_{\mathrm{aug}}^{\mathrm{T}})^{-1}$. The determinant $\left|\boldsymbol{\Sigma}_{\mathrm{aug}}\boldsymbol{\Sigma}_{\mathrm{aug}}^{\mathrm{T}}\right|$ and the cofactor matrix $\mathrm{adj}(\boldsymbol{\Sigma}_{\mathrm{aug}}\boldsymbol{\Sigma}_{\mathrm{aug}}^{\mathrm{T}})$ are shown in (29), (30), respectively.

$$\left|\boldsymbol{\Sigma}_{\mathrm{aug}}\boldsymbol{\Sigma}_{\mathrm{aug}}^{\mathrm{T}}\right| = \frac{\tau^6}{4M^4}\left\{\sum_{k=0}^{N-1}k^2\sum_{k=0}^{N-1}k^4\cdot N+2\sum_{k=0}^{N-1}k\sum_{k=0}^{N-1}k^2\sum_{k=0}^{N-1}k^3-\left(\sum_{k=0}^{N-1}k^2\right)^3-\left(\sum_{k=0}^{N-1}k\right)^2\sum_{k=0}^{N-1}k^4-\left(\sum_{k=0}^{N-1}k^3\right)^2\cdot N\right\} \tag{29}$$

$$\mathrm{adj}(\boldsymbol{\Sigma}_{\mathrm{aug}}\boldsymbol{\Sigma}_{\mathrm{aug}}^{\mathrm{T}}) = \begin{bmatrix}\frac{\tau^2}{M^2}\left(\sum_{k=0}^{N-1}k^2\cdot N-\left(\sum_{k=0}^{N-1}k\right)^2\right) & -\frac{\tau^3}{2M^2}\left(\sum_{k=0}^{N-1}k^3\cdot N-\sum_{k=0}^{N-1}k\sum_{k=0}^{N-1}k^2\right) & \frac{\tau^3}{M^4}\left(\sum_{k=0}^{N-1}k\sum_{k=0}^{N-1}k^3-\left(\sum_{k=0}^{N-1}k^2\right)^2\right)\\ -\frac{\tau^3}{2M^2}\left(\sum_{k=0}^{N-1}k^3\cdot N-\sum_{k=0}^{N-1}k\sum_{k=0}^{N-1}k^2\right) & \frac{\tau^4}{4M^2}\left(\sum_{k=0}^{N-1}k^4\cdot N-\left(\sum_{k=0}^{N-1}k^2\right)^2\right) & -\frac{\tau^5}{4M^3}\left(\sum_{k=0}^{N-1}k\sum_{k=0}^{N-1}k^4-\sum_{k=0}^{N-1}k^2\sum_{k=0}^{N-1}k^3\right)\\ \frac{\tau^2}{M^2}\left(\sum_{k=0}^{N-1}k\sum_{k=0}^{N-1}k^3-\left(\sum_{k=0}^{N-1}k^2\right)^2\right) & -\frac{\tau^5}{4M^3}\left(\sum_{k=0}^{N-1}k\sum_{k=0}^{N-1}k^4-\sum_{k=0}^{N-1}k^2\sum_{k=0}^{N-1}k^3\right) & \frac{\tau^6}{4M^4}\left(\sum_{k=0}^{N-1}k^2\sum_{k=0}^{N-1}k^4-\left(\sum_{k=0}^{N-1}k^3\right)^2\right)\end{bmatrix} \tag{30}$$

Finally, $\boldsymbol{\Sigma}_{\mathrm{aug}}^{\mathrm{T}}\left(\boldsymbol{\Sigma}_{\mathrm{aug}}\boldsymbol{\Sigma}_{\mathrm{aug}}^{\mathrm{T}}\right)^{-1}$ is derived in (31).

$$\begin{aligned}\boldsymbol{\Sigma}_{\mathrm{aug}}^{\mathrm{T}}\left(\boldsymbol{\Sigma}_{\mathrm{aug}}\boldsymbol{\Sigma}_{\mathrm{aug}}^{\mathrm{T}}\right)^{-1} &= \begin{bmatrix}\frac{(N-1)^2\tau^2}{2M} & \frac{(N-1)\tau}{M} & 1\\ \frac{(N-2)^2\tau^2}{2M} & \frac{(N-2)\tau}{M} & 1\\ \vdots & \vdots & \vdots\\ 0 & 0 & 1\end{bmatrix}\cdot\frac{4M^4}{\tau^6}\frac{1}{\left\{\sum_{k=0}^{N-1}k^2\sum_{k=0}^{N-1}k^4\cdot N+2\sum_{k=0}^{N-1}k\sum_{k=0}^{N-1}k^2\sum_{k=0}^{N-1}k^3-\left(\sum_{k=0}^{N-1}k^2\right)^3-\left(\sum_{k=0}^{N-1}k\right)^2\sum_{k=0}^{N-1}k^4-\left(\sum_{k=0}^{N-1}k^3\right)^2\cdot N\right\}}\\ &\cdot\begin{bmatrix}\frac{\tau^2}{M^2}\left(\sum_{k=0}^{N-1}k^2\cdot N-\left(\sum_{k=0}^{N-1}k\right)^2\right) & -\frac{\tau^3}{2M^2}\left(\sum_{k=0}^{N-1}k^3\cdot N-\sum_{k=0}^{N-1}k\sum_{k=0}^{N-1}k^2\right) & \frac{\tau^3}{M^4}\left(\sum_{k=0}^{N-1}k\sum_{k=0}^{N-1}k^3-\left(\sum_{k=0}^{N-1}k^2\right)^2\right)\\ -\frac{\tau^3}{2M_f^2}\left(\sum_{k=0}^{N-1}k^3\cdot N-\sum_{k=0}^{N-1}k\sum_{k=0}^{N-1}k^2\right) & \frac{\tau^4}{4M^2}\left(\sum_{k=0}^{N-1}k^4\cdot N-\left(\sum_{k=0}^{N-1}k^2\right)^2\right) & -\frac{\tau^5}{4M^3}\left(\sum_{k=0}^{N-1}k\sum_{k=0}^{N-1}k^4-\sum_{k=0}^{N-1}k^2\sum_{k=0}^{N-1}k^3\right)\\ \frac{\tau^2}{M^2}\left(\sum_{k=0}^{N-1}k\sum_{k=0}^{N-1}k^3-\left(\sum_{k=0}^{N-1}k^2\right)^2\right) & -\frac{\tau^5}{4M^3}\left(\sum_{k=0}^{N-1}k\sum_{k=0}^{N-1}k^4-\sum_{k=0}^{N-1}k^2\sum_{k=0}^{N-1}k^3\right) & \frac{\tau^6}{4M^4}\left(\sum_{k=0}^{N-1}k^2\sum_{k=0}^{N-1}k^4-\left(\sum_{k=0}^{N-1}k^3\right)^2\right)\end{bmatrix}\end{aligned} \tag{31}$$

By substituting (23) into (31), (32) is obtained.

$$\begin{aligned}q_{i1}&=\frac{M}{\tau^2(N-2)(N-1)N(N+1)(N+2)}\left\{360(N-i)^2+360(N-1)(N-i)-60(N-2)(N-1)\right\}\\ q_{i2}&=\frac{-M}{\tau(N-2)(N-1)N(N+1)(N+2)}\left\{180(N-1)(N-i)^2-12(2N-1)(8N-11)(N-i)+18(N-2)(N-1)(2N-1)\right\}\\ q_{i3}&=\frac{1}{N(N+1)(N+2)}\left\{30(N-i)^2-18(2N-1)(N-i)+3(3N^2-3N+2)\right\}\end{aligned} \tag{32}$$